Trigonometric Interpolation

A very brief review of Fourier Series.

Let $f \in C_p^m[0, 2\pi]$, $f : \mathbb{R} \to \mathbb{C}$. f has a Fourier Series representation

$$\boxed{f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}} \qquad (1).$$

Using orthogonality of the Fourier basis functions

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ikx} (e^{imx})^* dx = \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-m)x} dx$$

$$= \delta_{k-m} \qquad \left(\begin{array}{l} \text{~~Inner p~~ } L^2 \text{ inner product} \\ \langle f, g \rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x) g^*(x) dx \end{array} \right)$$

$$\cancel{\Rightarrow \frac{1}{2\pi}\int f(x) e^{-imx} dx}$$

$$\Rightarrow \boxed{\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx} dx} \qquad (2).$$

0.) f real $\Rightarrow f_k^* = f_{-k}$

1.) The decay of the amplitudes $\hat{f}_k$ is determined by the smoothness of f.

Let $\|f^{(m)}\|_\infty \leq L$

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(x) \frac{d}{dx}\left(\frac{e^{-ikx}}{-ik}\right) dx \quad , |k| > 0$$

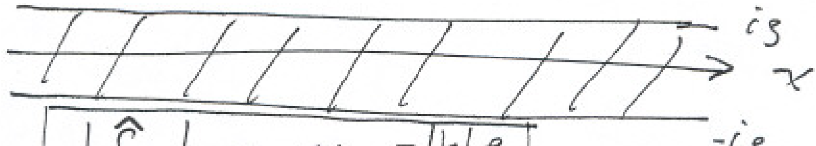
$$= \frac{1}{ik} \frac{1}{2\pi}\int_0^{2\pi} f'(x) e^{-ikx} dx$$

$$= \left(\frac{1}{ik}\right)^m \frac{1}{2\pi}\int_0^{2\pi} f^{(m)}(x) e^{-ikx} dx$$

$$\boxed{|\hat{f}_k| \le \frac{L}{|k|^m}} \qquad (3).$$

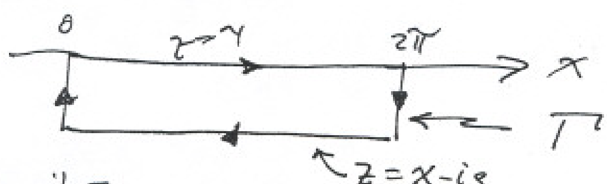
2.) If $f \in C_p^\infty[0, 2\pi]$, the Fourier amplitudes decay faster that any algebraic power of k.

3.) If f is analytic in a strip $[-i\rho, i\rho]$



$$\boxed{|\hat{f}_k| \le C e^{-|k|\rho}} \qquad (4)$$

Consider $k > 0$



end contributions cancel through periodicity.

$$0 = \frac{1}{2\pi} \int_\Gamma e^{ikz} f(z)\, dz$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\,dx - \frac{1}{2\pi}\int_0^{2\pi} e^{-ik(x-i\rho)} f(x-i\rho)\,dx$$

$$\Rightarrow \hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-k\rho} f(x-i\rho)\,dx$$

$$\Rightarrow |\hat{f}_k| \le e^{-k\rho} C'$$

For $k < 0$, use upper contour.

<u>Thm</u> The only $C_p^\infty[0,2\pi]$ ftns that satisfy $\|f^{(p)}\|_\infty \le L$, $\forall p$ with L indep. of p are proportional to 1, $\cos x$, & $\sin x$.

<u>Pf.</u> $\|f^{(p)}\|_\infty^2 \geq \frac{1}{2\pi}\int_0^{2\pi} |f^{(p)}(x)|^2 dx$

$$= \sum_k |k|^{2p} |\hat{f}_k|^2$$

Assume $|\hat{f}_{\bar{k}}| > 0$ for some $|\bar{k}| > 1$.

Then $\|f^{(p)}\|_\infty^2 \geq |\bar{k}|^{2p} C' \to \infty$ as $p \to \infty$.

Discrete Fourier Tf's (DFT)

Let $f_p = f(ph)$, $p = 0(1)(N-1)$, $h = 2\pi/N$, with N even. $\underset{\sim}{f} = \{f_p\}$

$$\boxed{f_p = \sum_{k=-\frac{N}{2}+1}^{N/2} \tilde{f}_k e^{ikph}} \quad (5)$$

i.e., there are N f_k's. $k = \frac{N}{2}$ is called the <u>Nyquist</u> <u>Frequency</u>.

- System (5) can be inverted for $\tilde{f}_k$ with a cost proportional to N^3, $(\mathcal{O}(N^3))$, using say Gaussian elimination.

Instead, use orthogonality of the discrete Fourier basis ftns

$$\underset{\sim}{E}_k = \{e^{ikph}\}_{p=0}^{N-1}, \quad k = (-\tfrac{N}{2}+1)(1)\tfrac{N}{2}.$$

under the inner product

$$\langle \underset{\sim}{f}, \underset{\sim}{g} \rangle_N = \frac{1}{N} \sum_{p=0}^{N-1} f_p g_p^* .$$

$$\langle E_k, E_m \rangle_N \qquad -\frac{N}{2}+1 \le k, m \le \frac{N}{2}$$

$$= \frac{1}{N} \sum_{p=0}^{N-1} e^{ikph} e^{-imph} = \frac{1}{N} \sum_{p=0}^{N-1} e^{iph(k-m)}$$

<u>Case 1:</u> $k - m = lN$, l integer

$$e^{iph(k-m)} = e^{ip \cdot \frac{2\pi}{N} \cdot lN} = 1$$

$$\Rightarrow \langle \underset{\to}{E}_k, \underset{\to}{E}_m \rangle_N = 1$$

Note: $l = 0$ is the only allowable for

$$-\frac{N}{2}+1 \le k, m \le \frac{N}{2}$$

<u>Case 2:</u> $k - m = j \neq lN$, l integer

$$\langle E_k, E_m \rangle_N = \frac{1}{N} \sum_{p=0}^{N-1} e^{ipjh} = \frac{1}{N} \sum_{p=0}^{N-1} (e^{ijh})^p$$

$$= \frac{1}{N} \frac{1-(e^{ijh})^N}{1-e^{ijh}} , \qquad e^{ijh} \neq 1 \text{ by assumption}$$

$$= \frac{1}{N} \frac{1-e^{ij\frac{2\pi}{N} \cdot N}}{1-e^{ijh}}$$

$$= 0.$$

$$\Rightarrow \boxed{\langle E_k, E_m \rangle_N = \delta_{k-m}}$$

$$\Rightarrow \langle \underset{\to}{f}, \underset{\to}{E}_m \rangle = \frac{1}{N} \sum_{p=0}^{N-1} f_p e^{-imph}$$

$$= \frac{1}{N} \sum_{p=0}^{N-1} e^{-imph} \left\{ \sum_{k=-\frac{N}{2}+1}^{N/2} \tilde{f}_k e^{ikph} \right\}$$

$$= \sum_{k=-\frac{N}{2}+1}^{N/2} \tilde{f}_k \left(\frac{1}{N} \sum_{p=0}^{N-1} e^{ikph} e^{-imph} \right)$$

$$= \sum_{k=-\frac{N}{2}+1}^{N/2} \tilde{f}_k \, \delta_{k-m} = \tilde{f}_m$$

And so we have

$$\boxed{\tilde{f}_k = \frac{1}{N} \sum_{p=0}^{N-1} f_p e^{-ikph}} \quad (6)$$

$$k = \left(-\frac{N}{2}+1\right)(1)\,\frac{N}{2}$$

0.) Now the cost is proportional to N^2 to find the N $\tilde{f}_k$'s. FFT is $\mathcal{O}(N \ln N)$

1/2.) (6) is the trap. rule approx to (2) for per. fns.

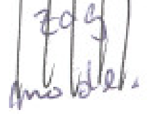
1.) f real

i.) $-\frac{N}{2}+1 \le k \le \frac{N}{2}-1$

$$\tilde{f}_{-k} = \tilde{f}_k^{*}$$

ii) $k = N/2$

Sawtooth mode.



$$\tilde{f}_{\frac{N}{2}} = \frac{1}{N} \sum_{p=0}^{N-1} f_p e^{-i\frac{N}{2}p \cdot \frac{2\pi}{N}} = \frac{1}{N} \sum_{p=0}^{N-1} f_p \cos p\pi$$

$$= \frac{1}{N} \sum_{p=0}^{N-1} (-1)^p f_p$$

$$\tilde{f}_{\frac{N}{2}} = \frac{1}{N}\sum_{p=0}^{N-1}(-1)^p f_p \quad , \quad \underline{\text{real for } f \text{ real.}}$$

This suggests the symmetric representation

$$f_p = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \tilde{F}_k e^{ikph}$$

with $\tilde{F}_k = \tilde{f}_k \qquad -\frac{N}{2}+1 \le k \le \frac{N}{2}-1$

and $\tilde{F}_{\pm\frac{N}{2}} = \frac{1}{2}\tilde{f}_{\frac{N}{2}}$

How is $\tilde{f}_k$ related to $\hat{f}_k$? The <u>aliasing error</u>.

$$f_p = f(ph) = \sum_{k=-\infty}^{\infty}\hat{f}_k e^{ikph}$$

$$\tilde{f}_m = \langle \underset{\rightarrow}{f}, \underset{\rightarrow}{E}_m \rangle_N = \frac{1}{N}\sum_{p=0}^{N-1}\left(e^{-imph}\sum_{k=-\infty}^{\infty}\hat{f}_k e^{ikph}\right)$$

$$= \sum_{k=-\infty}^{\infty}\hat{f}_k\left(\frac{1}{N}\sum_{p=0}^{N-1}e^{iph(k-m)}\right)$$

Recall $\frac{1}{N}\sum_{p=0}^{N-1}e^{iph(k-m)} = \begin{cases} 1, & k-m=lN \\ 0, & \text{o.w.}\end{cases}$

$$\boxed{\tilde{f}_k = \hat{f}_k + \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty}\hat{f}_{k+lN}} \qquad (7)$$

<u>The Aliasing Error.</u>

The DFT amplitudes are <u>precisely</u> related to the Fourier tf. amplitudes.

The error in $\tilde{f}_k$ is given by the smoothness of f.

i.) Let $f \in C_p^m[0, 2\pi]$, $\|f^{(m)}\|_\infty \leq L$

$$\Rightarrow |\hat{f}_k| \leq \frac{C}{|k|^m}, \quad |k| > 0.$$

$$|\tilde{f}_k - \hat{f}_k| \leq \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} |f_{k+lN}|, \quad -\frac{N}{2}+1 \leq k \leq \frac{N}{2}$$

$$\leq C \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \frac{1}{|k+lN|^m} = \frac{C}{N^m} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \frac{1}{|l+\frac{k}{N}|^m}$$

$$|l + \tfrac{k}{N}| \geq |l| - |\tfrac{k}{N}| \geq |l| - \tfrac{1}{2}$$

$$\leq \frac{2C}{N^m} \sum_{l=1}^{\infty} \frac{1}{(l-\frac{1}{2})^m}$$

$$\leq \frac{C_m}{N^m}$$

Note Algebraic decay w. speed of decay controlled by smoothness of f.

ii.) Let f be analytic in $[-i\rho, i\rho]$

$$\Rightarrow |\hat{f}_k| \leq C e^{-|k|\rho}$$

$$|\tilde{f}_k - \hat{f}_k| \leq C \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} e^{-|k+lN|\rho} \leq C\, 2 \sum_{l=1}^{\infty} e^{-N\rho(l-\frac{1}{2})}$$

$$\leq 2C e^{\frac{\rho N}{2}} \sum_{l=1}^{\infty} e^{-\rho N l} = 2C \frac{e^{-\rho N/2}}{1-e^{-\rho N}} \to 0$$

as $N \to \infty$.

Some applications of the DFT.

1.) Interpolation between mesh points.

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$$

$$P_f(x) = \sum_{k=-\frac{N}{2}}^{N/2} \tilde{F}_k e^{ikx}$$

Use the symmetric form to keep reality between mesh points.

<u>Or</u> $P_f(x) = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} \tilde{f}_k e^{ikx}$. Drop $k = N/2$. Retain reality, but ~~looks now no exact interpolation. Now have~~ no longer have exact interpolation. The error is small, & again depends on the smoothness of f.

2.) Differentiation & Integration

$$P_f'(x) = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} ik \tilde{f}_k e^{ikx}$$

$$P_{f,p}' = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} (ik\tilde{f}_k) e^{ikph}$$

~~Calculate $\tilde{f}_k$ & multiply by ik~~

$$\int_0^x P_f(x')\,dx' = x\tilde{\hat{f}}_0 + \sum_{\substack{k=-\frac{N}{2}+1 \\ k\neq 0}}^{\frac{N}{2}-1} \left(\frac{\tilde{\hat{f}}_k}{ik}\right)\left(e^{ikx}-1\right)$$

3.) Error Analysis of the Trapezoidal rule:

<u>$k=0$:</u> $\hat{f}_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$

$$\tilde{f}_0 = \frac{1}{N}\sum_{p=0}^{N-1} f_p = \frac{1}{2\pi}\, h \sum_{p=0}^{N-1} f_p$$

$f \in C_p^m[0, 2\pi]$

$$\Rightarrow \left|\hat{f}_0 - \tilde{f}_0\right| \leq \frac{C_m}{N^m}$$

See this again in the E-M formula.

4.) Analysis of linear difference methods.

Periodic

$S = C^2$ piecewise cubic interpolant

Cubic <u>Spline Eqns:</u>

$$\frac{h}{3}\left\{S'_{p-1} + 4S'_p + S'_{p+1}\right\} = f_{p+1} - f_{p-1}$$

$$f_p = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \tilde{f}_k e^{ikph}, \quad S'_p = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \tilde{S}'_k e^{ikph}$$

$$\Leftrightarrow \frac{h}{3} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \tilde{S}_k' \left(e^{ik(p+1)h} + 4 e^{ikph} + e^{ik(p-1)h} \right)$$

$$= \sum_{k=-\frac{N}{2}+1}^{N/2} \tilde{f}_k \left(e^{ik(p+1)h} - e^{ik(p-1)h} \right)$$

$$\Leftrightarrow \sum_{k=-\frac{N}{2}+1}^{N/2} \frac{h}{3} \tilde{S}_k' \left(2 \cos kh + 4 \right) e^{ikph}$$

$$= \sum_{k=-\frac{N}{2}+1}^{N/2} \tilde{f}_k \, 2i \sin kh \; e^{ikph}$$

Orthogonality gives

$$\Rightarrow \boxed{\tilde{S}_k' = \frac{3i \sin kh}{h(\cos kh + 2)} \tilde{f}_k}$$

$$= \frac{3 \sin kh}{kh(\cos kh + 2)} \; ik \tilde{f}_k$$

$$= 3 \left. \frac{\sin \theta}{\theta(\cos \theta + 2)} \right|_{\theta = kh} \underbrace{(ik \tilde{f}_k)}_{\text{DFT derivative.}}$$

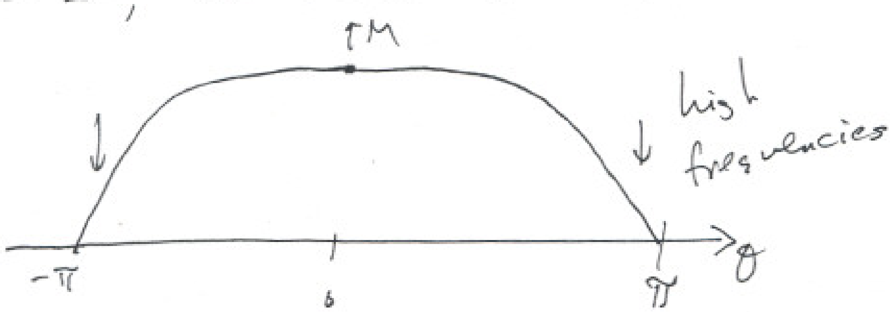
Note that $-\pi < \theta \le \pi$

$$\tilde{S}_k' = M(\theta = kh) \, (ik \tilde{f}_k)$$

$$\Rightarrow S_p' = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} M(kh)(ik \tilde{f}_k) \, e^{ikph}$$

, i.e. we're "diagonalized" the equation by the DFT.

If $M \equiv 1$, we have the DFT derivative



The spline derivative is the DFT derivative with a give premultiplier.

$$M(\theta) = 3\frac{\sin\theta}{\theta(\cos\theta + 2)}$$

Called a low-pass filter. Cuts off high-frequencies i.e. smooths!

1.) For $\theta \ll 1$, $M(\theta) = 1 + A\theta^4 + O(\theta^6)$

$\cancel{M(\theta) = 1 + A\theta^{P} \; O(\theta^{P}}$

$M(\theta) = 1 + A\theta^P$ gives a scheme w. accuracy $O(h^P)$.

2.) $\cancel{M(\pm\pi) = 0}$

The high frequencies in the DFT derivative are smoothed.

Convolution Thm

$$h(x) = \frac{1}{2\pi}\int_0^{2\pi} f(y)\, g(x-y)\, dy = (f*g)(x)$$

$$\frac{1}{2\pi}\int_0^{2\pi} dx\, h(x) e^{-ikx} = \hat{h}_k = \frac{1}{2\pi}\int_0^{2\pi} dx\, e^{-ikx} \times \frac{1}{2\pi}\int_0^{2\pi} dy\, f(y) g(x-y)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} dy\, f(y) \frac{1}{2\pi}\int_0^{2\pi} dx\, e^{-ikx} g(x-y)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} dy\, f(y)\, e^{-iky}\int_0^{2\pi} dx \frac{1}{2\pi} e^{-ik(x-y)} g(x-y)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} dy\, f(y) e^{-iky}\ \frac{1}{2\pi}\int_0^{2\pi} dz\, e^{-ikz} g(z)$$

$$= \hat{f}_k \cdot \hat{g}_k$$

DCT

$$\boxed{\begin{aligned} h_p &= \frac{1}{N}\sum_{j=0}^{N-1} f_j \cdot g_{p-j} \\ \tilde{h}_k &= \tilde{f}_k \cdot \tilde{g}_k \end{aligned}}$$

Many, many problems ~~involv~~ involve integrals of this form.

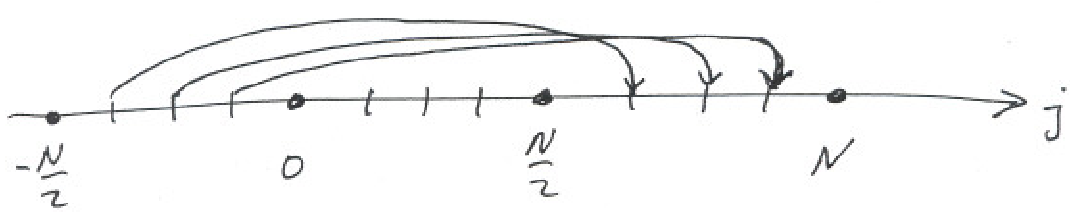
Why is this important? Naive evaluation is $\mathcal{O}(N^2)$. If we have an $\mathcal{O}(N \ln N)$ algorithm for eval. $\tilde{f}_k, \tilde{g}_k \rightarrow h_p$ very big savings.

The Fast Fourier Transform (FFT) algorithm

$$\begin{cases} f_k = \displaystyle\sum_{j=-\frac{N}{2}+1}^{N/2} \tilde{f}_j \, e^{ijkh}, & 0 \le k \le N-1 \qquad (1) \\[2ex] \tilde{f}_j = \dfrac{1}{N} \displaystyle\sum_{k=0}^{N-1} f_k \, e^{-ijkh}, & -\frac{N}{2}+1 \le j \le N/2 \qquad (2) \end{cases}$$

Define $\beta_j = \begin{cases} \tilde{f}_j, & 0 \le j \le N/2 \\ \tilde{f}_{j-N}, & \frac{N}{2}+1 \le j \le N-1 \end{cases}$ (3)

$0 \le j \le N-1$



$$f_k = \sum_{j=-\frac{N}{2}+1}^{-1} \tilde{f}_j \underbrace{e^{i(j+N)kh}}_{e^{ijkh} \cdot e^{i2\pi k}} + \sum_{j=0}^{N/2} \tilde{f}_j \, e^{ijkh}$$

$l = j+N$

$$= \sum_{l=\frac{N}{2}+1}^{N-1} \tilde{f}_{l-N} \, e^{ilkh} + \sum_{j=0}^{N/2} \tilde{f}_j \, e^{ijkh}$$

$$= \sum_{j=0}^{N-1} \beta_j \, e^{ijkh}$$ Typical presentation

$$\beta_j = \tilde{f}_j = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-ijkh}, \quad 0 \le j \le \frac{N}{2}$$

$$\beta_j = \tilde{f}_{j-N} = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i(j-N)kh}, \quad \frac{N}{2}+1 \le j \le N-1$$
$$= \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-ijkh}$$

$\Rightarrow$ The <u>symmetric form</u> for the DFT pair:

$$f_k = \sum_{j=0}^{N-1} \beta_j e^{ijkh} \qquad (4)$$

$$\beta_j = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-ijkh} \qquad (5)$$

Eqs. (4) & (5) are the usual representation used when discussing the FFT.

<u>Note:</u> $\bar{f}_k = \sum_{j=0}^{N-1} \bar{\beta}_j e^{-ijkh}$

$\Rightarrow$ The same algorithm is used to evaluate the DFT (~~given~~ $\beta_j \to f_k$) and the IDFT ($f_k \to \beta_j$).

Cooley - Tukey Algorithm

Let $N = 2^n$, $N = 2M$

Contrary to popular belief, FFT alg.s are not ~~resp~~ restricted to $N = 2^n$, i.e. powers of 2.

Phase polynomial:

$$p(x) = \sum_{k=0}^{N-1} \beta_k e^{ikx}$$

$p(x)$ interpolates f at $x_j = jh$

$$p(x_j) = f_j, \quad j = 0(1)(N-1)$$

Let $g(x)$ & $r(x)$ be phase polynomials satisfying

$$\begin{cases} g(x_{2j}) = f_{2j} \\ r(x_{2j}) = f_{2j+1} \end{cases} \qquad j = 0(1)(M-1)$$

$g(x)$ interpolates grid points of even index.

$\hat{r}(x) = r(x-h) = r(x - \pi/M)$ interpolates points of odd index.

Note: $e^{iMjh} = e^{iMx_j} = e^{ij\pi} = \begin{cases} 1, & j \text{ even} \\ -1, & j \text{ odd.} \end{cases}$

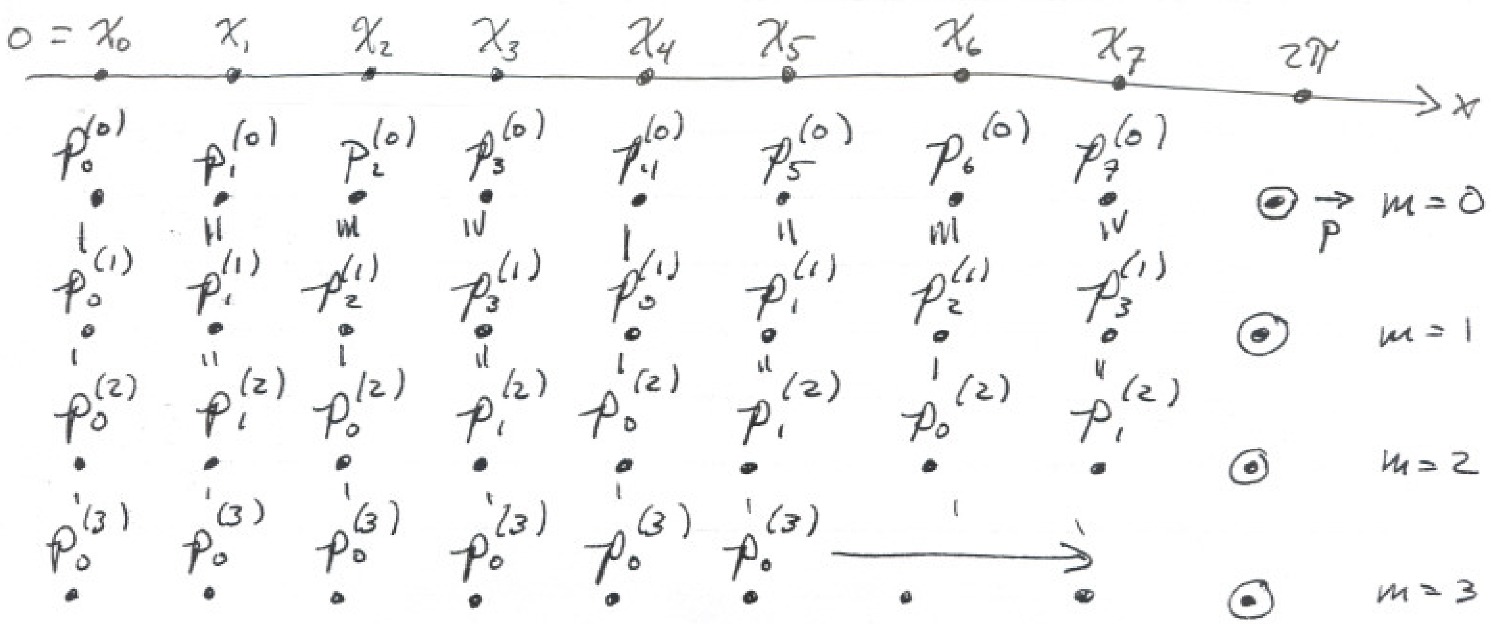
$$\Rightarrow \quad 2\,p(x) = \left(1+e^{iMx}\right) q(x) + \left(1-e^{iMx}\right) r\left(x-\frac{\pi}{M}\right) \tag{6}$$

i.e. $p(x)$ has been re-expressed in terms of lower-order phase polynomials.

Eq. (6) can be used iteratively to build up the entire interpolating phase polynomial.

<u>Ex:</u> $N=8$. At the 0^{th} step, we begin with 8 interpolating phase polynomials,

$$p_r^{(0)}\ , \quad r = 0(1)8 \quad \text{with } p_r^{(0)} = f_r$$



$m = 1$, 4 phase polynomials
$\quad 2$, 2 "
$\quad 3$, 1 "

$m = \ln_2 N$ steps.

An n-step recursion:

\# of points $2M$ | \# of phase polynomials

At the m^{th} step $(m \le n)$, let $M = 2^{m-1}$, $R = 2^{n-m}$

Determine R phase polynomials $\boxed{2R \cdot M = 2^n}$

$$p_r^{(m)}(x) = \sum_{k=0}^{2M-1} \beta_{r,k}^{(m)} e^{ikx}, \quad r = 0(1)(R-1)$$

from the $2R$ phase polynomials $p_r^{(m-1)}(x)$, $r = 0(1)(2R-1)$, by the recursion

$$\boxed{2p_r^{(m)}(x) = p_r^{(m-1)}(x)\left(1+e^{iMx}\right) + p_{r+R}^{(m-1)}\left(x-\frac{\pi}{M}\right)\left(1-e^{iMx}\right)}$$

$$\Rightarrow 2\sum_{j=0}^{2M-1} \beta_{r,j}^{(m)} e^{ijx} \left[= 2\sum_{j=0}^{M-1} \beta_{r,j}^{(m)} e^{ijx} + 2\sum_{j=0}^{M-1} \beta_{r,j+M}^{(m)} e^{i(j+M)x}\right]$$

$$= \left(\sum_{j=0}^{M-1} \beta_{r,j}^{(m-1)} e^{ijx}\right)\left(1+e^{iMx}\right)$$

$$+\left(\sum_{j=0}^{M-1} \beta_{r+R,j}^{(m-1)} e^{ij(x-\pi/M)}\right)\left(1-e^{iMx}\right)$$

$$= \sum_{j=0}^{M-1}\left(\beta_{r,j}^{(m-1)} + \beta_{r+R,j}^{(m-1)} e^{-ij\pi/M}\right) e^{ijx}$$

$$+ \sum_{j=0}^{M-1}\left(\beta_{r,j}^{(m-1)} - \beta_{r+R}^{(m-1)} e^{-ij\pi/M}\right) e^{i(j+M)x}$$

$\Rightarrow$ For $r = 0(1)(R-1)$, $j = 0(1)(M-1)$

$$2\beta_{r,j}^{(m)} = \beta_{r,j}^{(m-1)} + \beta_{r+R,j}^{(m-1)} \varepsilon_m^j$$

$$2\beta_{r,j+M}^{(m)} = \beta_{r,j}^{(m-1)} - \beta_{r+R,j}^{(m-1)} \varepsilon_m^j \qquad ; \ \varepsilon_m = e^{-i2\pi/2^m}$$

$2M \cdot R = 2^n = N$ opns

Begin the recursion with N phase poly's with

$$\beta_{k,0}^{(0)} = f_k \quad , \ k = 0(1)(N-1)$$

End with

$$\beta_j = \beta_{0,j}^{(n)} \quad , \ j = 0(1)(N-1)$$

Operation Count: $n \cdot N = N \ln_2 N$

$O(N \ln_2 N)$ opns. cheap.