Light interacting with liquid crystals

David W. McLaughlin a,1, David J. Muraki a,2 and Michael J. Shelley b,3

a Department of Mathematics, Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544, USA
b The Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA

In this paper we describe laser light interacting with nematic liquid crystals. The paper begins with a summary of recent experimental results of E. Braun, L. Faucheux, and A. Libchaber in which the liquid crystal sample is studied in three geometries – film, pipe, and droplet. Then, after a very brief glimpse at the history of liquid crystals, a theoretical model of the interacting system is described. In a one transverse dimensional idealization, we investigate the pipe and film configurations. In these cases the model reduces to a coupled system of nonlinear pde’s – an elliptic sine-Gordon equation for the director field coupled to a Schroedinger equation for the electromagnetic field. Properties and qualitative behavior of this coupled system are described, both numerically and theoretically. As an illustrative example of boundary layer analysis of such coupled light–nematic systems, we describe calculations in the film geometry in some detail. Results of this analysis include: (i) an extension of the Frederiks bifurcation analysis to electric fields with spatial variation; (ii) the determination of the transverse scale at which self-focusing saturates in this nematic; (iii) the derivation of a nonlocal nonlinear Schroedinger equation which governs the inner structure of the laser beam. We conclude the paper with a summary of similar boundary layer calculations for light–nematic systems in other geometries.

1. Introduction

Liquid crystals provide an optical medium with very strong nonlinearity. For example, the coefficient of nonlinearity in liquid crystals can easily be $10^6$–$10^{10}$ times greater than in “typical” nonlinear optical media such as carbon bisulfide (CS₂). The extreme nonlinearity of liquid crystals permits the investigation of nonlinear optical effects such as self-focusing and filamentation with low power cw lasers. For this reason liquid crystals are excellent materials for the investigation of laser radiation interacting nonlinearly with matter, and thus they provide an optical arena for the study of the physics of complex nonlinear patterns.

In recent experiments of E. Braun, A. Libchaber, and L. Faucheux [1], nematic liquid crystals (MBBA, 6CB, and E209) were irradiated with light from a cw argon laser with 1 watt of power. A linearly polarized argon laser beam in the $TEM_{00}$ gaussian mode (the pump beam) was applied to the sample of liquid crystal which sat on the stage of an inverted microscope. The pump beam had a waist of $\approx 50 \mu m$ at the bottom plate of the sample. Thermal effects were negligible, while scattering losses were more substantial but controlled.

A detailed description of the experimental results may be found in [2,1]. Here we restrict ourselves to a few remarks about them, focusing attention upon two representative experiments. Studies were performed in three geometries – film, droplet, and capillary tube. Self-focusing effects were observed in all three geometries. In

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1 This lecture is dedicated to Al Scott, who has shown us many beautiful properties of the sine-Gordon equation.
2 Funded in part by AFOSR-90-0161 and by NSF DMS 8922717 A01.
the thin film case, the observations were made outside the liquid crystal "after" the light beam had passed through the thin sample; thus, self-focusing was observed only indirectly. However, in the droplet and tube cases, filamentation was directly observed. Figure 1 shows a beautiful example of focusing and filamentation in the droplet geometry.

In order to achieve better control of thermal effects, Braun, et al. [2,1] changed to the geometry of a cylindrical capillary tube, together with a different nematic liquid crystal, E209, which offers a wide range in temperature in the nematic phase. A cylindrical capillary tube, of inside diameter $\approx 1.5$ mm, was coated with the polymer MAP to enforce tangential alignment of the director at the glass surface, and thus to ensure uniform boundary conditions along the capillary tube. The liquid crystal was then injected in the nematic phase. One end of the capillary was sealed with fast epoxy to avoid any motion of the nematic. Cooling was achieved by a flow of nitrogen around the capillary. The strongest lens in this system is a diverging one due to the meniscus at the air–liquid crystal interface. With this setup, the argon laser beam was focused inside the sample, far from the entrance meniscus, and the scattered light was observed from below.

Figure 2 shows a striking longitudinal view of the light field along the capillary tube. Many
other observations, for several increasing values of power in the pump beam, can be found in [2,1]. There one observes the presence of a focal spot, and, as the pump intensity is increased, its "movement" along the tube toward the meniscus at the front entry. As the intensity is further increased, this movement stops and a second spot forms and moves toward the front of the tube. Three coexisting focal spots have been observed in the experiments [2,1]. Also one observes the onset of transverse undulations of the filament. The wavelength of these undulations at first decreases, and then increases with increasing pump intensity [1]. Finally, at still higher values of pump intensity, two filaments form and interact. As far as we know, these are the first observation of the actual longitudinal behavior of filamentation of a low power cw laser beam in a nonlinear medium. In any case, it is certainly a striking visualization.

2. Brief overview of the history of liquid crystals

The twelfth International Liquid Crystal Conference was held in Germany in 1988. One aspect of that conference was the celebration of the one hundredth anniversary of the "discovery" of liquid crystals. At that conference, H. Kelker described that history. Our account is taken from the second chapter of a recent book by Peter Collings [3].

During the time period from 1850-1888, just before the "discovery" of liquid crystals, several European scientists were observing striking visual effects such as cloudiness and color changes which would later be attributed to the liquid crystal phase of their samples. For the most part these scientists were biologists who were investigating under microscopes the effect of temperature on different biological materials including the outer covering of nerve fibers, natural fats, and cholesterol. An Austrian botanist, Friedrich Reinitzer, is generally credited with the actual discovery of liquid crystals in his studies of cholesterol in plants. In 1888 he was observing the melting behavior of an organic substance related to cholesterol and noted that it possessed two melting points - at 145.5°C it melted to a cloudy liquid and at 178.5°C this cloudy liquid turned clear. He also observed the appearance of blue colors at the phase transitions. The substance that Reinitzer was observing was in fact cholesteryl benzoate, a chiral nematic liquid crystal.

Reinitzer sent some of his samples to a German physicist, Otto Lehmann, who had just developed an important microscope. Lehmann's microscope was a significant technical advancement for that time period in that it had a heating stage that allowed him to observe crystallization as temperature was slowly lowered, as well as polarizers which allowed him to view polarization phenomena. Through his observations of Reinitzer's samples, Lehmann became convinced that the cloudy liquid was a liquid phase, but a phase that affected polarized light in a manner typical of solid crystals. The substance had the flow properties of a liquid, but the optical properties of a solid, and led Lehmann to call such substances "liquid crystals".

As described by Collings [3], European chemists, biologists, and physicists continued to study liquid crystals, both experimentally and theoretically, during the first half of the twentieth century. The theory culminated with the continuum elastic theory of English physicist F. C. Frank in the 1950's. Little was done in liquid crystal research from the end of World War II through the early 1960's, perhaps because no important applications had emerged. The discovery of liquid crystal displays (LCD's) at the end of the 1960's certainly changed the status of research in the area! Today liquid crystal physics is alive, well, and dynamic – with a great many scientists actively studying nonlinear properties of this liquid crystal phase of matter.

While this "history of liquid crystals" is certainly interesting, we elected to describe it in such detail in our paper for another reason. Our
conference here at the Technical University of Denmark has as one of its goals the celebration of the sixtieth birthday of Professor Alwyn C. Scott. As we learned about the history of liquid crystals, we were struck by the similarities between the development of the science of liquid crystals and the manner with which AI does, and advocates doing, science: - The development of liquid crystals was interdisciplinary – between the biological and physical sciences; international; nonlinear, and driven by experimental, theoretical, and practical developments. A brief consideration of such historical perspectives certainly seems appropriate at our conference on Future Directions of Nonlinear Dynamics in Physical and Biological Systems.

We conclude this historical section with two additional references: the book of de Gennes [4], and the survey [5].

3. Theoretical considerations

We are interested in a coupled field description of the interaction of laser light with a nematic liquid crystal. A nematic liquid crystal may be thought of as consisting of cigar shaped "objects" which are either polarized or easily polarizable. These objects have both translational and orientational degrees of freedom. In the liquid crystal phase, the material behaves as a fluid with respect to its translational degrees of freedom; however, with respect to its orientational degrees of freedom, long range order is maintained as if the material were in a crystalline phase. In this work we will ignore fluid motion and concentrate upon the orientational behavior of the liquid crystal.

This orientation is described by a director field, \( \mathbf{n}(x, t) \), \(|\mathbf{n}| = 1\).

This unit vector \( \mathbf{n}(x, t) \) describes the density of "rods", at position \( x \) at time \( t \), oriented in the direction \( \mathbf{n} \).

Rotation of these rods is opposed by elastic, nearest neighbor, forces. Three independent orientational distortions exist for the vector field \( \mathbf{n}(x, t) \). These are called "twist", "splay", and "bend"; they are displayed in the cartoons of Fig. 3 [6]; the energy densities associated with each are given by

\[
\begin{align*}
\mathcal{E}_T &= K_T [\mathbf{n} \cdot (\nabla \times \mathbf{n})]^2, \\
\mathcal{E}_S &= K_S (\nabla \cdot \mathbf{n})^2, \\
\mathcal{E}_B &= K_B [\mathbf{n} \times (\nabla \times \mathbf{n})]^2,
\end{align*}
\]

where \( K_T, K_S, \) and \( K_B \) are the elastic constants associated with twist, splay, and bend respectively. The total elastic energy is then given by

\[
\mathcal{E}_{el} = \frac{1}{2} \int (\mathcal{E}_T + \mathcal{E}_S + \mathcal{E}_B) \, d^3x.
\]

Infinitesimal variations (subject to the unit director constraint) of this elastic energy produce an elliptic operator which, in the "one constant" approximation \( K \equiv K_S = K_B = K_T \), reduces to the Laplacian. (Typical nematic liquid crystals have \( K_S < K_B \sim K_T \).)

We seek to describe the interaction of this director field with laser light. First, note that the
director \( \mathbf{n} \) is very sluggish in time and can only respond to the time-averaged electromagnetic field \( \mathbf{E} \). A phenomenological description of this interaction is as follows: The liquid crystal has its director field \( \mathbf{n} \) specified on the boundary of the sample. When the laser shines on this sample, energy minimization favors the alignment of \( \mathbf{n} \) with \( \mathbf{E} \). During this alignment procedure, \( \mathbf{E} \) induces a dipolar change in the index of refraction. The index of refraction is maximal when \( \mathbf{n} \) is parallel to \( \mathbf{E} \), producing a "self-focusing" lens. The reason behind the large values of the nonlinear dielectric constant in liquid crystals is that it is relatively easy to rotate the director field.

The coupled equations for the electric field interacting with the director field are given by

\[
-\nabla \times \nabla \times \mathbf{E} + k_0^2 (\mathbf{E} + \alpha (\mathbf{n} \cdot \mathbf{E})\mathbf{n}) = 0,
\]

\[
K \left( \mathbf{n} \times \nabla^2 \mathbf{n} + g \Re (\mathbf{n} \times (\mathbf{n} \cdot \mathbf{E}) \mathbf{E}^*) \right) = 0,
\]

where \( g \) denotes a coupling parameter. Here we note that \( \mathbf{E} \) is the time averaged complex electric field amplitude, and that the interaction in the coupled equations (3.1) is invariant to \( \mathbf{n} \rightarrow -\mathbf{n} \).

It will be convenient to specialize to a simple two-dimensional geometry which can apply in either the "film" or the "tube" cases:

\[
\mathbf{n} = \begin{pmatrix} \sin \theta \\ 0 \\ \cos \theta \end{pmatrix}, \tag{3.2}
\]

\[
\mathbf{E} = \begin{pmatrix} F \\ 0 \\ H \end{pmatrix}, \tag{3.3}
\]

where the angle \( \theta = \theta(x, z) \), and the components of the field \( (F, H) \) also only depend upon \( (x, z) \). In this two-dimensional geometry, \( z \) is the longitudinal coordinate; \( x \) is the transverse coordinate; \( F \) and \( H \) are the transverse and longitudinal field components, respectively. If, at "leading order", one neglects the longitudinal field \( H \), the equation for the director takes the form of an elliptic, variable coefficient, sine-gordon equation:

\[
\theta_{xx} + \theta_{zz} + |F|^2 \sin 2\theta = 0. \tag{3.4}
\]

### 3.1. The Frederiks transition

The relevant nonlinear effect for our studies is known as the "Frederiks transition" and can be easily seen from eq. (3.4): One assumes an \( F \) field which is constant in both \( x \) and \( z \), and seeks a director field \( \theta \) which is a function of \( z \) only. In this situation, one reduces (3.4) to the ode

\[
\theta_{zz} + \lambda^{-2} \sin 2\theta = 0,
\]

\[
\theta(z = 0) = \theta(z = L) = 0.
\]

Here we have introduced a parameter \( \lambda \) known as the Frederiks transition length,

\[
\lambda \equiv |F|^{-1},
\]

and have introduced the boundary conditions on the director that \( \mathbf{n} = \mathbf{z} \) at \( z = 0 \) and \( z = L \); that is, \( \theta \) vanishes at the front and rear faces of the sample. \( \theta = 0 \) is one solution of this two-point boundary problem corresponding to alignment with the boundary conditions of the director throughout the sample. On the other hand, as the intensity of the constant \( F \) field is increased, solutions of the two-point boundary experience a "pitchfork bifurcation" and a second solution appears with \( \theta = \theta(z) \). This second solution represents a \( z \) dependent rotation of the director field within the sample, away from its boundary orientation toward the orientation of the \( x \) component of the electric field. This bifurcation is the primary nonlinear effect behind our study. We will investigate it numerically and analytically, in the more general situation of coupled fields with both \( x \) and \( z \) dependence.
3.2. A model problem

By scaling all lengths on the Frederiks transition length of the incoming pump beam, the experimental value of the optical wavenumber becomes \( k \approx 189 \). (Other natural scalings produce much larger nondimensional wave numbers.) With such a large wavenumber, the electric field is certain to obey a slowly varying envelope approximation. Here we study the simplest such (two dimensional) envelope model – one which incorporates a coupling between paraxial optics and nematic deformation:

\[
2ikF_z + F_{xx} + k^2 \alpha \sin^2 \theta F = 0, \tag{3.5}
\]

\[
\theta_{zz} + \theta_{xx} + |F|^2 \sin 2\theta = 0. \tag{3.6}
\]

Here the refractive anisotropy is represented by the parameter \( \alpha \), which is approximately 0.25 for the experiments [2,1]. Depending upon boundary conditions, this model can be used to study either the thin film or the tube geometries. In the thin film case, the boundary conditions are

\[
F(x, z = 0) = F_{in}(x)
\]

\[
\theta(x, 0) = \theta(x, L) = 0,
\]

\[
\theta(x, z) \to 0 \quad \text{as} \quad x \to \pm \infty. \tag{3.7}
\]

(In the tube geometry, the latter boundary condition is replaced by \( \theta(\pm W, z) = 0 \), where \( W \) is the width (radius) of the tube.)

Heuristically, eq. (3.5) is a scalar paraxial equation that includes the effects of diffraction and nematic anisotropy, while eq. (3.6) models static nematic distortions. Of course, this two dimensional model neglects birefringence (polarization) effects and scattering losses – which are important for quantitative detail. However, this simple model has proven useful in the development and description of our mathematical approximations.

4. An initial numerical study

We begin with a brief summary of our initial numerical study of model (3.5)–(3.6). To set up the numerical algorithm we impose Dirichlet boundary conditions for the \( \theta \) field in \( z \), Neumann boundary conditions for the \( \theta \) field in \( x \), as well as periodic boundary conditions with respect to the transverse variable \( x \) for the \( F \) field. The transverse width models an extremely wide thin film, and in practice is chosen wide enough that it does not effect the results. (The choice of transverse Neumann boundary conditions has been made to improve this insensitivity.)

The parabolic equation for \( F \) is integrated with an “integrating factor” method which exactly factors out the Laplacian from the evolution in \( z \) by introducing analytically the factor \( \exp \left( \frac{iz}{2k} \theta_{xx} \right) \). The elliptic equation for the \( \theta \) field is integrated with a “relaxation method”. Both the “integrating factor” and “relaxation” methods are implemented through “fast Fourier transforms”, which are natural given the boundary conditions.

Sample results from these numerical experiments are displayed in Fig. 4. In these numerical experiments, \( k = 100.00, \alpha = 0.10 \), and the critical field for the “Frederiks transition” (below which there is no transition) is \( |F_{cr}| \approx 0.12 \). In Fig. 4a, the maximum field intensity is \( |F_{max}| = 0.20 \), somewhat above the transition value; in Fig. 4b, the maximum field intensity is larger, \( |F_{max}| = 0.50 \), well above the transition value. Examination of these figures shows:

(i) The transverse profile of the electric field \( |F| \) is much narrower than that of the nematic field \( \theta \).

(ii) The contour lines of the nematic field \( \theta \) show transition regions (in \( z \)) near the front and rear faces of the nematic film. These rather sharp transitions are particularly apparent at the larger input field of Fig. 4b.

(iii) As the electric field \( F \) “propagates” in \( z \), it “self-focuses” in that its intensity increases while its transverse structure narrows. Again,
Fig. 4. Sample numerical experiments for the scalar model problem, for $\alpha = 0.10, k = 100.00$. At these parameter values the critical field for the Frederiks transition has $|F_{c1}|^2 \approx (0.12)^2$. (a) Pump beam: $|F_{\text{max}}|^2 = (0.2)^2$. (b) Pump beam: $|F_{\text{max}}|^2 = (0.5)^2$. In each case, we display contour lines for the nematic angle $\theta$; the electric field intensity profiles $|F|^2$ vs $x$, at the front (dotted curve) and rear (solid curve) faces; the behavior of the electric field intensity (solid) and of the nematic (dotted) along the center beam line; and a transverse profile near the rear face for the electric field intensity (solid) and nematic angle (dotted).
this self-focusing is particularly apparent for the larger input field of Fig. 4b. In both cases a and b, the self-focusing continues to the rear face of the film. It is not clear from these experiments what, if anything in the absence of scattering losses, saturates this self-focusing process.

Clearly, for this simple model problem, these numerical discretization effects could be eliminated with a finer resolution of the numerical grid. Furthermore, one could also further stress the model by running the numerical experiments at more realistic parameter values—for example, \( k \) could be increased from 100.00 to 189.00 and \( \alpha \) from 0.1 to 0.25. However, the “boundary-layer–self-focusing” nature of the phenomena make this solely numerical approach unrealistic for the actual system. Thus, even for the model problem, it is judicious to develop a mathematical analysis based upon a boundary layer reduction.

5. A boundary layer analysis for the thin film case

With our thin film model (3.5), (3.6), (3.7), we illustrate a boundary layer approach for the light–nematic system. The geometry is depicted in Fig. 5. We emphasize that the boundary layer used in the analysis is a transverse boundary layer located at the beam center. While the nematic field \( \theta \) has contour lines (see Fig. 4) reminiscent of longitudinal boundary layer behavior near the front and rear faces of the nematic film, this longitudinal transition scale is much longer than the characteristic width of the self-focused electric field \( F \). In the absence of scattering losses, the self-focusing continually narrows the electric field until diffraction saturates this process at a very narrow beam. This transverse self-focusing is also apparent in the numerics of Fig. 4b. Indeed, it is this transverse effect of the electric field, rather than the longitudinal behavior of the nematic, that limits numerical resolution, and introduces the small width scale for our analysis.

Let \( \gamma = \gamma(k) \) denote a large parameter which fixes the inner, transverse beam scale:

\[
\tilde{x} \equiv \gamma x, \quad \gamma \gg 1.
\]

Here the scaling function \( \gamma(k) \) will be determined later in the analysis; for the moment it is only taken to be large. Next we assume that in the outer-relaxation and in the inner-beam regions, the fields take the following form:

\[
\theta \simeq \begin{cases} 
\theta^{(r)}(z,x) & \text{(outer)}, \\
\theta^{(l)}(z,0) + \gamma^{-1}\theta^{(b)}(z,\tilde{x}) & \text{(inner)},
\end{cases}
\]

(5.1)

\[
F \simeq \begin{cases} 
0 \text{ (exp. small)} & \text{(outer)}, \\
\gamma^{1/2}F^{(b)}(z,\tilde{x}) & \text{(inner)}.
\end{cases}
\]

(5.2)

When viewed from the outer scale, the \( |F|^2 \) field appears as a delta function located at the beam center \( x = 0 \); in turn, from eq. (3.6), the \( \theta \) field has a jump discontinuity in its first derivative \( \theta_x(x = 0, z) \). Thus,

\[
|F|^2 \simeq I \delta(x),
\]

where

\[
I \equiv \int_{-\infty}^{+\infty} |F|^2 \, dx = \int_{-\infty}^{+\infty} |F^{(b)}|^2 \, d\tilde{x},
\]
and where the scaling of $F^{(b)}$ insures the last equality.

In the two outer-relaxation regions ($x > 0$ and $x < 0$), the electric field is exponentially small and the nematic field satisfies Laplace's equation

$$\nabla^2 \theta^{(r)} = 0,$$

together with vanishing boundary conditions at $z = 0, z = L, x = \pm \infty$ and the jump condition across $x = 0$. Thus, $\theta$ has the Fourier series representation

$$\theta^{(r)} = \sum_{n=1}^{\infty} \hat{c}_n \exp \left( -n \frac{\pi x}{L} \right) \sin \left( n \frac{\pi z}{L} \right),$$

where the constants $\hat{c}_n$ will be determined by the jump condition. Notice that because $\theta^{(r)}$ solves Laplace's equation, the transverse half-width of the nematic distortion is determined by the thickness $L$ of the thin film.

To derive the jump condition, one integrates eq. (3.6) across the beam zone, paying attention to scales and to the local behavior of the fields. The result is

$$\theta^{(r)}_{x=0^+} + I \sin 2\theta^{(r)}_{x=0^-} = 0.$$

Defining

$$C(z) \equiv \theta^{(r)}(z, x=0) = \sum_{n=1}^{\infty} \hat{c}_n \sin \left( n \frac{\pi z}{L} \right),$$

one sees that the jump condition yields the expression

$$\frac{I}{2} \sin \left[ 2 \sum_{n=1}^{\infty} \hat{c}_n \sin \left( n \frac{\pi z}{L} \right) \right]$$

$$= \sum_{n=1}^{\infty} \hat{c}_n \sin \left( n \frac{\pi z}{L} \right),$$

which is a nonlinear equation which determines the constants $\hat{c}_n$ as a function of the beam intensity $I$; thus, it determines the nematic distortion at beam center as described by the function

$$C(z, I).$$

This equation (5.3) should be viewed as an extension of the Frederiks bifurcation equation to the boundary layer situation of variable field strength. It must be solved numerically. Its solution will behave as a pitch-fork bifurcation and, above a critical intensity $I_c$, will have the qualitative behavior sketched in Fig. 6. Note, in particular, this nematic distortion $C(z, I)$ contains the "longitudinal transitions" in $z$.

Turning to the behavior of the electric and nematic fields in the inner-beam region, we express the nematic equation (3.6) in the inner coordinates and solve it asymptotically for $\theta^{(b)}$,

$$\theta^{(b)} = -\frac{i}{2} \sin (2C) \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F|^2 \, dy \, dy \right)$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F|^2 \, dy \, dy.$$

Inserting this expression into the field equation (3.5) written in inner variables yields, after removing an appropriate fast phase $kS(x, z)$,

$$iG_x + \frac{\gamma^2}{2k} G_{xx} + \frac{k\alpha}{2\gamma} \sin (2C) \theta^{(b)} G = 0,$$

where

$$F^{(b)} = G \exp (ikS).$$
Examination of eq. (5.5) shows that the natural balance of diffraction against nematic distortion defines the scaling

$$y(k) = k^{2/3}. \quad (5.6)$$

With this choice of $y$ scaling, and introducing the propagation scale $\tilde{z} \equiv k^{1/3}z$, we obtain the leading order equation which governs the structure of the inner beam:

$$iG_x + \frac{1}{2}G_{xx} - \frac{1}{4} \alpha \sin^2(2C) \left\{ \int_{-\infty}^{\infty} |G|^2 \, dy \right\} G = 0. \quad (5.7)$$

Several remarks should be made about this nonlocal, nonlinear Schrödinger equation (5.7) which governs the local structure of the inner electric field:

(i) This nonlinear Schrödinger equation is a Hamiltonian system, with Hamiltonian

$$H = \int_{-\infty}^{\infty} \left\{ \frac{1}{2} |G_x|^2 + \frac{1}{4} \alpha \sin^2(2C) \left( \int_{-\infty}^{\infty} |G|^2 \, dy \right) \right\} \, dx. \quad (5.8)$$

(ii) It depends upon $C(z)$, the nematic distortion at beam center, which is determined from the outer problem as matched to the inner beam via bifurcation equation (5.3).

(iii) Even though this NLS equation has coefficients which depend upon $z$, the intensity $I$ is independent of $z$ and can be viewed as a parameter which is controlled by the input laser beam. However, perhaps the most important consequence of this boundary layer analysis follows from the combination of the inner beam equation (5.7) with the scaling law (5.6). Together, these imply that, in the absence of scattering losses, the self-focusing process terminates on the scale $x \approx O(y^{-1} = k^{-2/3}).$

6. Conclusion

In this paper we have described some initial work on the interaction of a laser beam with a nematic liquid crystal. Our interest in this system stems from its very large coefficient of nonlinearity which allows the generation of complex nonlinear optical patterns with low power, CW lasers. Experimentally, Braun, Faucheux, and Libchaber have studied this system in three geometric configurations of film, droplet, and tube [2], [1]. Of particular importance in these experimental studies are the striking longitudinal views of interacting filaments in the cylindrical geometry.

In this paper we have described a theoretical boundary layer analysis of an idealized (two dimensional, scalar) model of this coupled light-nematic system in a thin film geometry. The transverse boundary layer analysis shows that the nematic distortion at beam center is described by a “local Frederiks bifurcation curve” which captures transitions in the nematic near the front and rear faces of the film. The analysis also produces a scaling function which results from “maximal balance” between diffraction and nonlinearity and fixes (in the absence of scattering losses) the spatial extent of the self-focussed beam. It also yields a nonlocal nonlinear Schrödinger equation which governs the inner structure of the beam.

In other theoretical boundary layer analyses of this coupled system, we have studied the two dimensional “tube geometry” for both a model scalar [1] and full vector [7] optical field. In [1] we show that in the tube geometry, the two dimensional scalar model captures the essential physics of both the filamentation and the transverse undulation of the laser beam. The latter is a new phenomenon. In order to describe polarization effects, which are certainly absent in these scalar models, we have studied a full (but two dimensional) vector system in the tube geometry [7]. In this study we identify properties of the undulation process that are distinctly polar-
ization effects.

Taken together, these experimental, numerical, and theoretical studies establish that the light–nematic system is an excellent source of complex, nonlinear spatial patterns. Furthermore, our studies show that coupled nonlinear pde descriptions of light–nematic interactions are indeed accessible theoretically, through boundary layer asymptotics implemented numerically.

Acknowledgements

We gratefully acknowledge Erez Braun, Luc Faucheux, and Albert Libchaber, who have introduced us to liquid crystal self-focusing through their elegant experiments, and with whom we have enjoyed many hours of intense collaboration.

References