Shear instability for stratified hydrostatic flows

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Abstract

Stratified flows in hydrostatic balance are studied in both their multilayer and continuous formulations. A novel stability criterion is proposed for stratified flows, which re-interprets stability in terms not of growth of small perturbations, but of the well-posedness of the time evolution. This re-interpretation allows one to extend the classic results of Miles and Howard concerning steady and planar flows, to the realm of flows that are non-uniform and unsteady.

1 Introduction

Stratified flows occur ubiquitously in nature, with the atmosphere and ocean as prime examples. When the horizontal scales of a flow are much larger than the vertical ones, the flow satisfies to a good approximation the *hydrostatic balance*, whereby the pressure at each position balances the weight of the fluid above it. Models for hydrostatically balanced flows come in two main flavors: multilayered models, where the flow is assumed piecewise uniform in the vertical, and models with continuous shear and stratification. Typical real flows have a continuous stratification profile, but approximately layered flows do arise, for instance, in river outflows, and in the "staircase" stratification profiles often observed in the ocean [12]. In addition, many numerical discretizations of the continuous equations mimic the physics of discrete homogeneous layers.

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Layered models are systems of conservation laws for mass and momentum in each layer. Continuous models can also be written as systems of infinitely many conservation laws, corresponding to layers of vanishing thickness, or as an infinite system of evolution equations for vertical eigenmodes. Most stability analyses have adopted the latter approach whereas here we explore the former. This is best achieved through the introduction of an isopycnal coordinate system [1], whereby the fluid's density ρ replaces the depth z as vertical coordinate. (This system is appropriate for incompressible fluids; the equivalent formulation for compressible flows involves isentropic coordinates. For brevity, we will only concern ourselves here with incompressible fluids.)

This article uses the mathematical machinery behind systems of conservation laws to shed light on issues pertaining stratified flows, both discrete (layered) and continuous. In particular, it re-interprets the stability properties of the system in terms of the well-posedness of its time evolution, instead of the more customarily used rate of growth of small perturbations. This corresponds to classifying the system according to its type: hyperbolic corresponding to stability (more precisely, well-posedness), and elliptic corresponding to instability (ill-posedness). Similar characterizations of stability in terms of type have been noticed in rheological problems; see, for instance, [3].

This view provides a useful extension of stability theorems, such as the classical results of Miles and Howard [13, 4], from their planar scenarios to more general, non-uniform and unsteady flows. In the unstable Miles-Howard scenario, arbitrarily large growth rates (hence the ill-posedness) occur for small perturbations of sufficiently short wave-lengths. This feature is captured by a re-interpretation in terms of a hyperbolic-elliptic transition. The physical manifestation of this transition remains the standard one: the time evolution becomes mathematically ill-posed because it is no longer possible to neglect mixing, as the model does.

Finally, solutions that remain in the hyperbolic regime can be nevertheless unstable in the classical sense: small perturbations may grow, though the growth rates are bounded, and the catastrophic scenario of high frequency perturbations growing arbitrarily fast does not arise. In fact, examples of this scenario, with unsteady flows with Richardson numbers bigger than $\frac{1}{4}$ but are nonlinearly unstable over long space and time scales, are presented in [10, 11].

This article exploits the strong analogy between the equations for layered and continuously stratified flows. Section 2 presents a derivation of the corresponding models in parallel, attempting to achieve maximal simplicity. One can factor out the mean stratification profile (or layer thickness) from the equations, allowing solutions that consist of vertically periodic variations superimposed onto a background stratification.

Section 3 characterizes and computes the simple waves of the system [8]. As building blocks, simple waves have the advantage over the more standard linear modes, that they constitute fully nonlinear solutions. In particular, they break, hence leading to another mechanism for fluid mixing. To our knowledge, simple waves in systems of infinitely many conservation laws have not been considered before.

Section 4 introduces the characterization of stability based on equation type, and uses it to extend the Miles-Howard theorem on shear stability from steady planar flows to more general non-uniform, unsteady profiles. It also develops similar constructions for layered flows.

2 Layered and continuous stratified flows

In this section the equations describing hydrostatically balanced stratified flows are derived for multilayer systems, and continuously stratified flows are then computed as a limiting case. In the Boussinesq approximation, one can factor out the mean stratification. This allows considering flows that are a vertically periodic perturbation of a background profile.

In a system of N layers with uniform density and horizontal velocity, conservation of mass for each layer reads

$$h_t^j + (h^j u^j)_x = 0\,, \tag{2.1}$$

where h^j is the layer thickness and u^j the fluid velocity. Conservation of momentum adopts the form

$$\begin{aligned}(\rho^j h^j u^j)_t + \left(\rho^j h^j (u^j)^2 + \frac{1}{2}(p^{j+\frac{1}{2}} + p^{j-\frac{1}{2}})(z^{j+\frac{1}{2}} - z^{j-\frac{1}{2}})\right)_x &= \\ = p^{j+\frac{1}{2}}\left(z^{j+\frac{1}{2}}\right)_x - p^{j-\frac{1}{2}}\left(z^{j-\frac{1}{2}}\right)_x\,, \end{aligned}\tag{2.2}$$

where ρ^j is the density of layer j, and p and z stand for the pressure and height at the interfaces between layers. The terms on the right-hand side of the momentum equation represent the form drag among layers. Since this is an internal force of the system, it dissappears when adding over all layers, yielding the conservation of total momentum

$$\left(\sum_1^N \rho^j h^j u^j\right)_t + \left(\sum_1^N h^j \left(\rho^j (u^j)^2 + \frac{1}{2}(p^{j+\frac{1}{2}} + p^{j-\frac{1}{2}})\right)\right)_x = 0 \tag{2.3}$$

The pressure satisfies the hydrostatic relation

$$p^{j-\frac{1}{2}} - p^{j+\frac{1}{2}} = g\rho^j h^j\,, \tag{2.4}$$

where g is the gravity constant. Using conservation of mass and the hydrostatic balance to simplify the momentum equation (2.2), one obtains

$$u_t^j + u^j u_x^j + \frac{1}{\rho^j}\left(\frac{p^{j+\frac{1}{2}} + p^{j-\frac{1}{2}}}{2} + g\rho^j\,\frac{z^{j+\frac{1}{2}} + z^{j-\frac{1}{2}}}{2}\right)_x = 0\,.$$

In terms of the Montgomery potential

$$M^j = \frac{1}{2}(p^{j+\frac{1}{2}} + p^{j-\frac{1}{2}}) + g\rho^j \frac{1}{2}(z^{j+\frac{1}{2}} + z^{j-\frac{1}{2}}), \tag{2.5}$$

the system can be written in the form

$$\begin{aligned} h_t^j + (h^j u^j)_x &= 0, \\ u_t^j + u^j u_x^j + \frac{1}{\rho^j} M_x^j &= 0, \\ \frac{M^{j+1} - M^j}{\rho^{j+1} - \rho^j} &= g z^{j+\frac{1}{2}}, \\ z^{j+\frac{1}{2}} - z^{j-\frac{1}{2}} &= h^j \end{aligned}$$

The Boussinesq appoximation involves neglecting the effects of density changes on the inertia, that is, replacing ρ^j by a constant ρ_0 in the equation for u_t^j above. Then, assuming for simplicity that $\Delta\rho = \rho^j - \rho^{j+1}$ is independent of j, and non-dimensionalizing

$$\begin{aligned} \frac{\sqrt{g' h_0}}{x_0} t &\to t \\ \frac{x}{x_0} &\to x \\ \frac{u^j}{\sqrt{g' h_0}} &\to u^j \\ \frac{M^j}{g \Delta\rho\, h_0} &\to M^j \\ \frac{h^j}{h_0} &\to h^j \end{aligned}$$

(where $g' = g\frac{\Delta\rho}{\rho_0}$ is the reduced gravity, x_0 is a typical length, and the thickness h_0 is defined for later convenience as $h_0 = \frac{1}{N}\sum_1^N h^j$), one obtains the set of equations describing multilayer flows:

$$\begin{aligned} h_t^j + (h^j u^j)_x &= 0, \\ u_t^j + u^j u_x^j + M_x^j &= 0, \\ \Delta_2 M^j &= -h^j, \end{aligned} \tag{2.6}$$

where Δ_2 is the discrete second difference

$$\Delta_2 M^j = M^{j+1} - 2M^j + M^{j-1}.$$

In order to obtain the equations describing continuous stratification, one uses the alternative non-dimensionalization

$$\begin{aligned}
\sqrt{\frac{g\,\delta}{\rho_0\Delta\rho}}\,\frac{t}{x_0} &\to t \\
\frac{x}{x_0} &\to x \\
\sqrt{\frac{\rho_0\Delta\rho}{g\,\delta}}\,u^j &\to u^j \\
\frac{\Delta\rho\,M^j}{g\,\delta} &\to M^j \\
\frac{h^j}{\delta\,\Delta\rho} &\to h^j\,,
\end{aligned}$$

where $\delta = \frac{1}{r}\int h\,d\rho$ and r is the density difference between the bottom and top of the domain. Taking the limit of small $\Delta\rho$, yields the system of partial differential equations modeling internal waves in isopycnal coordinates:

$$\begin{aligned}
h_t + (hu)_x &= 0, \\
u_t + uu_x + M_x &= 0, \\
M_{\rho\rho} &= -h.
\end{aligned} \tag{2.7}$$

These equations are strongly reminiscent of those describing a single layer of shallow water flow, but with the pressure term M and the layer thickness h, which are identical in shallow waters, related instead through a Poisson problem in the vertical.

Subtracting from both the continuous and discrete systems the mean stratification (or mean layer width), one can make the replacements

$$\begin{aligned}
h^j &\to \left(1 - S^j\right) \\
M^j &\to M^j - \frac{1}{2}j(j-1)\,,
\end{aligned}$$

for the discrete system and, for the continuously stratified case,

$$\begin{aligned}
h &\to (1 - S) \\
M &\to M - \frac{1}{2}\rho^2\,.
\end{aligned}$$

It is not necessary to assume constant background stratification –or layer width– to make these substitutions: the variables $S^j(x)$ may have a nonzero horizontal average. Their vertical average, on the other hand, is zero, due to our choice of h_0

and δ in the nondimensionalizations above. The discrete equations become

$$\begin{aligned} S^j{}_t - \left(\left(1-S^j\right)u^j\right)_x &= 0 \\ u^j{}_t + u^j u^j{}_x + M^j{}_x &= 0\,, \\ \Delta_2 M^j &= S^j \end{aligned} \tag{2.8}$$

and the continuous ones

$$\begin{aligned} S_t - ((1-S)u)_x &= 0, \\ u_t + uu_x + M_x &= 0, \\ M_{\rho\rho} &= S. \end{aligned} \tag{2.9}$$

This setting permits the use of periodic boundary conditions in the vertical direction, which simplifies many theoretical developments. Hence, from now on, the dependent variables S, M and u, are periodic in x and in either j or ρ:

$$S^{j+N}(x,t) = S^j(x,t)\,, \quad S^j(x+L,t) = S^j(x,t)\,, \quad \ldots$$

$$S(x,t,\rho+r) = S(x,t,\rho)\,, \quad S(x+L,t,\rho) = S(x,t,\rho)\,, \quad \ldots$$

In addition, S has zero vertical mean:

$$\sum_1^N S^j = 0 \qquad \left[\int_0^r S(x,t,\rho)\,d\rho = 0\right],$$

which implies that

$$\sum_1^N \left(\left(1-S^j\right)u^j\right)_x = 0 \qquad \left[\int_0^r ((1-S)u)_x\,d\rho = 0\right]. \tag{2.10}$$

In fact, more is true: the volume flux

$$Q = \sum_1^N \left(1-S^j\right)u^j\,, \qquad \left[Q = \int_0^r (1-S)\,u\,d\rho\right]$$

is, under the Boussinesq approximation, also the total momentum (replace ρ^j by ρ_0 in the momentum density in equation (2.3)). Hence, it is not only spatially uniform, as (2.10) implies, but also constant in time, since total momentum is conserved. This can be shown from the equations in (2.8) by adding them by parts, which yields

$$\left[\sum_1^N \left(1-S^j\right)u^j\right]_t + \left[\sum_1^N \left(1-S^j\right){u^j}^2 + M^j + \frac{1}{2}(M^{j+1}-M^j)^2\right]_x = 0\,. \tag{2.11}$$

Equivalently, in the continuous case,

$$\left[\int (1-S)\,u\,d\rho\right]_t + \left[\int \left((1-S)u^2 + M + \frac{1}{2}\left(M_\rho\right)^2\right)d\rho\right]_x = 0\,. \tag{2.12}$$

Integrating these equations over the horizontal period, and using the fact -from (2.10)- that the term differentiated with respect to time is spatially uniform, it follows that the total volume flux (and momentum density) Q is constant [1].

One can apply this result in closing the systems (2.8) and (2.9). At first sight it is not entirely clear how the evolution equations in either of these systems are to be closed. The difficulty is that M is not entirely determined from S by the Poisson problem – since the vertical mean $\overline{M}$ of M is left undetermined. However, in view of (2.11) and (2.12), the terms differentiated in time dissappear (since they are constant), and we obtain

$$\overline{M}_x = -\left(\overline{\frac{1}{2}(\Delta M)^2 + (1-S)\,u^2}\right)_x \tag{2.13}$$

in the discrete case, and

$$\overline{M}_x = -\left(\overline{\frac{1}{2}M_\rho^2 + (1-S)\,u^2}\right)_x \tag{2.14}$$

in the continuous one.

Both systems (2.8) and (2.9) conserve energy, given by

$$E = \int (E_k + E_p)\,dx\,, \tag{2.15}$$

where

$$E_k = \sum_{j=1}^{N} \frac{1}{2}(1-S_j)\,u_j^2 \qquad \left[E_k = \int_0^r \frac{1}{2}(1-S)\,u^2\,d\rho\right] \tag{2.16}$$

and

$$E_p = \sum_{j=1}^{N} \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} \rho\,z\,dz = \sum_{j=1}^{N} \frac{1}{2} z_{j+\frac{1}{2}}^2 \qquad \left[E_p = \int \rho\,z\,dz = \int_0^r \frac{z^2}{2}\,d\rho\right] \tag{2.17}$$

are the kinetic and potential energy densities, respectively.

3 Simple waves

As the number of layers in the system grows, and in the continuous limit, the system's behavior becomes increasingly complex. It is convenient then to isolate individual simple waves, which can be thought of as "building blocks" of the more complex dynamics [8]. Moreover, as we shall see, the study of simple waves sheds light on the system's stability.

For a nonlinear hyperbolic system of the type

$$v_t + A(v)v_x = 0\,, \tag{3.1}$$

[1] In particular, if a Galilean transformation is performed so that Q is initially zero, it remains zero forever.

where $v(x,t)$ is a vector with components v_j, simple waves are solutions of the form

$$v_j(x,t) = V_j(\xi(x,t))\,. \tag{3.2}$$

In other words, all components of the vector solution are functions of a single scalar function $\xi(x,t)$. Plugging this ansatz into (3.1) yields the eigenvalue problem

$$(A(V(\xi)) - c(\xi)I)V_\xi = 0\,, \tag{3.3}$$

where

$$c(\xi) = -\frac{\xi_t}{\xi_x}\,,$$

which can be re-written as an equation for the evolution of ξ:

$$\xi_t + c(\xi)\xi_x = 0\,. \tag{3.4}$$

The characteristic speed $c(\xi)$ follows from the eigenvalue of (3.3); the corresponding eigenvector V_ξ yields by integration $V(\xi)$, the phase-space representation of the simple wave. Notice that there is some freedom in the determination of $V(\xi)$: the initial vector $V(\xi = 0)$ can be chosen arbitrarily, and the eigenvector V_ξ can be re-scaled at will (this second freedom, however, amounts to just a reparameterization of $V(\xi)$.) Once $c(\xi)$ is known, we can find $\xi(x,t)$ from any initial data $\xi(x,0)$ by solving the scalar equation (3.4) by the method of characteristics, and then reconstruct the full vector solution $v(x,t)$ through the identity (3.2).

Figures 1 and 2 show snapshots of two simple waves, one corresponding to the largest (first-baroclinic) eigenvalue of a three-layer system, and the other to the second-largest (second-baroclinic) eigenvalue of a sixteen-layer system. In both cases, sinusoidal initial data are followed as they evolve into a breaking wave. In order to continue the solution after the breaking time, a closure is required, possibly involving mixing [5, 15, 6].

The idea of simple waves can be extended from the multilayer case to the continuously stratified equations (2.9), by replacing the index j in (3.2) by the continuous variable ρ:

$$\begin{aligned} u(x,t,\rho) &= U(\xi(x,t),\rho) \\ M(x,t,\rho) &= M(\xi(x,t),\rho) \end{aligned} \tag{3.5}$$

(where we use M rather than S as dependent variable, so as to have differential rather than integral operators). This results in the system

$$\begin{pmatrix} (u-c)\frac{\partial^2}{\partial\rho^2} & M_{\rho\rho} \\ 1 & u-c \end{pmatrix} \begin{pmatrix} M_\xi \\ u_\xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.6}$$

which yields the second-order differential equation for M_ξ:

$$(u-c)^2 \frac{\partial^2 M_\xi}{\partial\rho^2} = M_{\rho\rho}\, M_\xi\,, \tag{3.7}$$

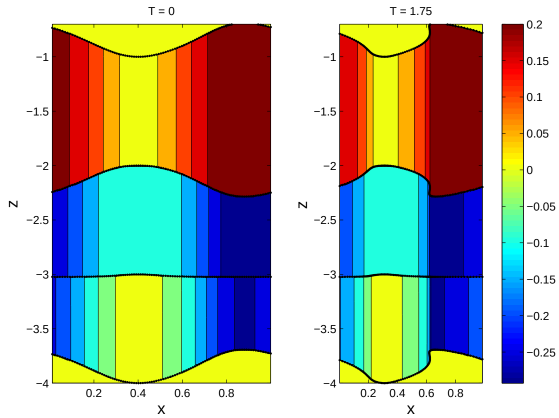

FIGURE 3.1. First baroclinic simple wave in a three-layer system, up to its breaking time. The color code represents the values of the velocity u, and the solid lines the position of the interfaces between layers. The figure on the left has the initial value, corresponding to a sinusoidal profile for $\xi(x,0)$; the figure on the left depicts the simple wave near its breaking time. The profile is vertically periodic: the form of the interface at the base of the bottom layer matches the one at the top of the uppermost layer.

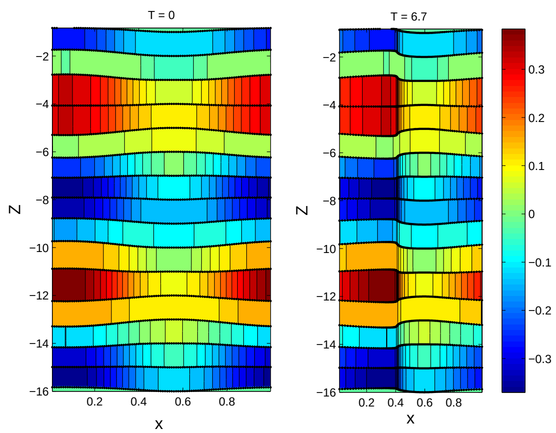

FIGURE 3.2. Second baroclinic simple wave in a sixteen-layer system, up to its breaking time. The conventions for plotting and the meaning of the two figures are the same as for Figure 1.

To our knowledge, this idea of extending simple waves to continuous systems of [infinitely many] conservation laws has not been explored before. For stratified flows, it allows us to go beyond the well-known linear internal waves [9] to their fully nonlinear counterpart.

4 Stability criteria

The eigenvalue problem (3.3) does not necessarily have only real solutions. In other words, stratified flows are systems of *mixed type*: hyperbolic when all eigenvalues of $A(v)$ are real, and elliptic otherwise. The presence of elliptic domains is associated with an instability of the system: when the equations turn elliptic, the initial-value problem for them becomes ill-posed [7], and solutions can be found that are initially arbitrarily close, yet diverge from each other in arbitrarily short time intervals. This perspective on the stability problem is more powerful than the conventional one, in which one looks for exponentially growing solutions of the equations linearized around a global profile. With this approach we can characterize the stability of non-uniform, evolving, solutions, and detect the places where the time-evolution becomes ill-posed, and hence mixing is needed. The calculation is local: it suffices to determine whether the system matrix at any particular point (x,t) has or not a complete set of real eigenvalues.

Ill-posedness is a much more dramatic scenario that regular instability. The fact that perturbations grow does not necessarily invalidate a model, nor calls for a new closure. Yet when the growth-rate of perturbations is unbounded, the model stops making sense. Hence ill-posedness is typically an indication that new physics is required. In the present study, the missing physics is that of mixing among fluid layers.

The classic results of Miles and Howard [13, 4] on shear-instability for stratified flows can be re-interpreted in terms of well-posedness, and hence extended to non-planar, unsteady profiles: When the eigenvalue c is complex, equation (3.7) can be rewritten in the form

$$\frac{\partial^2 M_\xi}{\partial \rho^2} = \frac{M_{\rho\rho}}{\left((u-c_r)^2+c_i^2\right)^2}(u-c_r+ic_i)^2 M_\xi \,.$$

Multiplying through by $\bar{M_\xi}$ and integrating in ρ, we obtain

$$\int \left|\frac{\partial M_\xi}{\partial \rho}\right|^2 + \frac{M_{\rho\rho}}{\left((u-c_r)^2+c_i^2\right)^2}(u-c_r+ic_i)^2 \left|M_\xi\right|^2 d\rho = 0\,.$$

The imaginary part of this equation yields

$$2c_i \int \frac{M_{\rho\rho}}{\left((u-c_r)^2+c_i^2\right)^2}(u-c_r) \left|M_\xi\right|^2 d\rho = 0\,.$$

Hence a necessary condition for the imaginary part of c not to vanish is that $(u-c_r)$ change sign; in particular, there must exist a critical layer where $c_r = u$. Near such point, one may expand (3.7) into

$$(u_\rho(\rho_0))^2(\rho-\rho_0)^2\frac{\partial^2 M_\xi}{\partial\rho^2}=M_{\rho\rho}(\rho_0)M_\xi\,,$$

or

$$(\rho-\rho_0)^2\frac{\partial^2 M_\xi}{\partial\rho^2}+Ri\,M_\xi=0\,, \tag{4.1}$$

where the Richardson number Ri is given by

$$Ri=\frac{M_{\rho\rho}(\rho_0)}{(u_\rho(\rho_0))^2}.$$

This Euler equation has solutions of the form

$$M_\xi=|\rho-\rho_0|^\alpha\,,$$

with

$$\alpha=\frac{1}{2}\pm\sqrt{\frac{1}{4}-Ri}\,.$$

When $Ri>1/4$, these solutions are singular, suggesting that for $Ri>1/4$ it is not possible for c to have an imaginary part. This fact can be established rigorously, following the same lines as in the classical proof by Miles [13] for shear-instability of planar flows. Instead, we mimic the alternative, simpler proof by L. Howard [4] in the paper immediately following Miles' in the same issue. Returning to equation (3.7), and making the change of variables

$$M_\xi=\left(\sqrt{u-c}\right)\,\phi,$$

equation (3.7) adopts the self–adjoint form

$$\left((u-c)\phi'\right)'+\left(\frac{1}{2}u''-\frac{M_{\rho\rho}+\frac{1}{4}u'^2}{u-c}\right)\phi=0\,.$$

Multiplying by $\bar{\phi}$ and integrating, one obtains

$$\int\left[-(u-c)|\phi'|^2+\left(\frac{1}{2}u''-\frac{M_{\rho\rho}+\frac{1}{4}u'^2}{u-c}\right)|\phi|^2\right]d\rho=0\,.$$

The imaginary part of this expression is

$$c_i\int\left[|\phi'|^2-\frac{M_{\rho\rho}+\frac{1}{4}u'^2}{|u-c|^2}|\phi|^2\right]d\rho=0\,. \tag{4.2}$$

It follows that, for c_i not to vanish,

$$Ri=-\frac{M_{\rho\rho}}{u'^2}<\frac{1}{4}\,. \tag{4.3}$$

Hence, for an hydrostatically balanced flow to be ill-posed (elliptic) it is necessary that the local Richardson number be smaller than one quarter, a result that extends the range of validity of Miles' and Howard's beyond steady, planar stratification profiles.

Similar computations can be carried out for multilayer flows. First, we prove that instability can only arise when there are critical layers. In the discrete case, this means that the real part of the unstable eigenvalue has to lie within the range of the velocities. To show this, the equation for simple waves (corresponding to (3.7) in the continuous case) is

$$\left[(u_j - c)\,\Delta_2 + \frac{1-S_j}{u_j - c}\right] M_{\xi,j} = 0.$$

Multiplying by $\overline{M}_{\xi,j}$ and summing by parts, one obtains

$$\sum_1^N \left|\Delta^+ M_{\xi,j}\right|^2 - \frac{1-S_j}{(u_j-c)^2}\left|M_{\xi,j}\right|^2 = 0\,.$$

The imaginary part is

$$-2c_i \sum_1^N (u_j - c_r)\frac{1-S_j}{|u_j-c|^4}\left|M_{\xi,j}\right|^2 = 0\,,$$

and, clearly if $c_i \neq 0$, then $u_j - c_r$ has to change sign, and hence c_r is in the range of the u_j.

To make an argument similar to Howard's in the discrete case, one introduces $\phi_j = \frac{M_{\xi,j}}{\sqrt{u_j-c}}$, multipllies (4) by $\frac{\overline{\phi}_j}{\sqrt{u_j-c}}$ and adds by parts, yielding

$$\sum_1^N \Bigg[\sqrt{(u_j-c)\,(u_{j+1}-c)}\left|\Delta^+\phi_j\right|^2 - \left(\frac{1-S_j}{u_j-c} + \sqrt{u_j-c}\,\Delta_2\sqrt{u_j-c}\right)\left|\phi_j\right|^2\Bigg] = 0\,,$$

where $c = c_r + ic_i$. The real part of c can be absorbed in the u_j's, and the imaginary part of the equation becomes

$$\sum_1^N Im\left[\sqrt{(u_j-ic_i)\,(u_{j+1}-ic_i)}\right]\left|\Delta^+\phi_j\right|^2 - \left(\frac{1-S_j}{u_j^2+c_i^2}\,c_i + Im\left[\sqrt{u_j-ic_i}\,\Delta_2\sqrt{u_j-ic_i}\right]\right)\left|\phi_j\right|^2 = 0\,,$$

or, if $c_i \neq 0$,

$$\sum_1^N F(\tilde{u}_j,\tilde{u}_{j+1})\left|\Delta^+\phi_j\right|^2 + \left(\frac{1-S_j}{c_i^2(\tilde{u}_j^2+1)} + 2 - F(\tilde{u}_j,\tilde{u}_{j+1}) - F(\tilde{u}_j,\tilde{u}_{j-1})\right)\left|\phi_j\right|^2 = 0\,,$$

where

$$\tilde{u}_j = \frac{u_j}{c_i}, \qquad F(u,v) = \frac{1}{\sqrt{2}}\sqrt{1 - uv + \sqrt{(1+u^2)\,(1+v^2)}}\,.$$

This expression can be simplified further by changing variables to $u = \sinh(x)$, $v = \sinh(y)$. Then $F(u(x), v(x)) = \cosh\frac{x-y}{2}$, and, in terms of

$$\theta_j = \sinh^{-1}\left(\frac{u_j}{c_i}\right), \tag{4.4}$$

one obtains

$$\sum_1^N \cosh\frac{\theta_j - \theta_{j+1}}{2}\left|\Delta^+\phi_j\right|^2 + \left(\frac{1-S_j}{c_i^2\cosh^2(\theta_j)} - 2\sinh^2\left(\frac{\theta_j - \theta_{j-1}}{4}\right) - 2\sinh^2\left(\frac{\theta_j - \theta_{j+1}}{4}\right)\right)\left|\phi_j\right|^2 = 0\,,$$

the discrete equivalent to (4.2).

In fact, if one considers the differences in θ's to be small and performs the corresponding truncated Taylor expansion, one recovers (4.2) exactly. However, it is not clear how to go beyond this heuristics, and obtain a necessary condition for instability as in the continuous case.

The problem is that the θ differences are not necessarily small even if those in the u's are: the occurrence of c_i in (4.4) amplifies, when c_i is small, the effect of the shear. Because of this difficulty, we analyze the stability of multilayer flows on a case by case basis in [14, 2], concentrating on two and three-layer systems. These are not only the cases that appear most frequently in applications, but also constitute, in a certain sense, the discrete equivalent of (4.1), the expansion near a critical layer: one is not allowed in the multilayer case to Taylor-expand the fields, but may instead attempt to capture the presumably "local" character of the instability, by concentrating on the layers between which the sign of $(u_j - c_i)$ changes.

5 Conclusions

This article formulates a unified description of stratified flows in hydrostatic balance, including both multilayered and continuously stratified flows. Removing the mean stratification profile from the independent variables allows one to consider vertically periodic fluctuations around a background stratification profile. These periodic flows provide a simpler setting for analytical considerations than flows bounded in the vertical by lids or free surfaces.

By adopting a multilayered–isopycnal approach, one can think of the equations as a system of conservation laws indexed by the density. This realization permits computing nonlinear simple waves up to their breaking time, and to introduce a novel characterization of stability, based on the local type (hyperbolic or elliptic) of the system. With this characterization, one can to re-interpret classical theorems on shear instability of stratified flows and extend them into the realm of non-uniform, unsteady flows.

The methodology and results described in this article form one point of departure for the study of mixing, the onset of which can be attributed frequently to shear instability and breaking waves.

Appendix

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