Leaky rigid lid: new dissipative modes in the troposphere

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ABSTRACT

An effective boundary condition is derived for the top of the troposphere, based on a wave radiation condition at the tropopause. This boundary condition, which can be formulated as a pseudo-differential equation, leads to new vertical dissipative modes. These modes can be computed explicitly in the classical setup of a hydrostatic, non-rotating atmosphere with a piecewise constant Brunt-Väisälä frequency.

In the limit of an infinitely strongly stratified stratosphere, these modes loose their dissipative nature and become the regular baroclinic tropospheric modes under the rigid-lid approximation. For realistic values of the stratification, the decay time-scales for the first few modes range from an hour to a week, suggesting that the time-scale for many atmospheric phenomena may be set up by the rate of energy loss through upwards propagating waves.

Introduction

Much of our understanding of tropospheric dynamics is based on the concept of discrete internal modes. The pressure disturbances that govern our weather, for instance, propagate at definite speeds, typically associated with the first to third baroclinic vertical modes, depending on the nature of the disturbance. Even though other effects such as nonlinearity, moist convection and mean wind shear alter significantly the nature and speed of these waves, they remain nonetheless the dynamical backbone of the troposphere.

Yet discrete modes are the signature of systems of finite extent: a semi-infinite stratified atmosphere yields a continuum spectrum of modes, much as the Fourier transform in the infinite line, as opposed to the discrete Fourier series associated with finite intervals. This has led to arguments by R. Lindzen that these discrete tropospheric modes are just a fallacy of overly simplified theoretical models, and that the atmosphere "is characterized by a single isolated eigenmode and a continuous spectrum" Lindzen (2003). On the other hand, the troposphere does seem to operate on distinct discrete modes, and many phenomena have been modeled successfully on such basis.

The simplest and most conventional way to obtain a discrete set of tropospheric modes with realistic values for speed and vertical structure is to replace the tropopause by a rigid lid where the vertical velocity must vanish. Two justifications are typically provided for this approximation. One is that, for internal baroclinic waves, the oscillations at the free-surface of stratified fluids have much smaller amplitude than those at internal isopycnals, as demonstrated in the famous experiment of Franklin with water and oil, and manifested in the dead water phenomenon Franklin (1905); Ekman and Bjerknes (1904). This is indeed the basis for the rigid-lid approximation for the surface of the ocean, widely used for the study of its internal dynamics. Yet the tropopause is not the free surface between two fluids of very different density: it is not the density but its vertical derivative that has a discontinuity at the interface.

The second justification, more appropriate in the atmospheric context, is that the stratosphere, being much more strongly stratified than the troposphere, inhibits vertical motion. Yet the ratio of the stratification of the stratosphere to that of the troposphere, as measured by their representative Brunt-Väisälä frequencies, is not infinite; in fact it is rather close to 2. Can the rigid-lid approximation be justified under these circumstances? Do new effects come into play due to this finite ratio? These are the questions addressed in this article. As we shall see, the answer is affirmative to both. The main new effects are: 1) the modes dissipate, as they radiate a fraction of their energy into the stratosphere, 2) a slight change in the speed and vertical structure of the modes, and 3) the appearance of a new tropospheric mode, with some barotropic characteristics and a dissipation time-scale close to an hour.

The need to impose boundary conditions at a finite height, such as the top of the troposphere or the upper end of a finite computational domain, has lead to a variety of modeling approaches. The simplest boundary condition for a model of a finite atmosphere is a rigid lid, which has the vertical velocity set to zero at some finite height. Even though it is not completely justified on sound physical grounds, this boundary condition gives rise to one of the fundamental tools for understanding atmospheric dynamics – the rigid lid modes. These have been used for a number of theoretical purposes, such as to study resonant interaction among waves Raupp et al. (2008), to identify wave activity in the observational record Haertel et al. (2008) and to study tropical–extratropical teleconnections Kasahara and da Silva Dias (1986). The rigid lid is also an essential part of some modeling strategies for introducing moist dynamics into atmospheric models, projecting the dynamics onto the first few baroclinic rigid lid modes to yield a simplified vertical structure of the atmosphere with minimal vertical resolution Majda and Shefter (2001). Prior attempts at improved boundary conditions relied on some form of radiation condition allowing all the internal gravity waves to leave the computational domain. For example, see Bennett (1975), Klemp and Durran (1983), Garner (1986), and Purser and Kar (2001). Some authors skip this issue altogether and model the atmosphere as infinite; for example, the MIT GCM uses pressure coordinates. Modeling the atmosphere as infinite versus finite can lead to mathematically drastically different solution properties; for example in the first case a continuous spectrum in the vertical direction may arise, while in the second case the spectrum may be fully discrete.

In this paper we introduce two new results. First, we derive an effective boundary condition (EBC) to be used at the top of the troposphere, so that it can be modeled in isolation from the rest of the atmosphere. Second, we present "leaky" rigid lid modes, which are computed using the EBC for a simple example background stratification. These modes have a novel feature: they decay with realistic time-scales.

To derive these results we perform a local calculation at the interface between the troposphere and the stratosphere, to obtain reflection and transmission coefficients. These coefficients characterize how much wave energy leaves the troposphere, and we use them next to construct the EBC. The most important assumption in our approach is that no waves return back from the stratosphere to the troposphere – though waves are allowed to reflect at the tropopause. As long as our assumption holds, we can substitute the stratosphere with the effective boundary condition. Ongoing research shows that our modeling approach can be extended to incorporate more complicated physics of the troposphere, such as the Earth's rotation and possibly convection and moisture. Yet in this article that introduces the leaky lid, we have purposely concentrated on the simplest scenario of dry irrotational linear waves in a non-rotating environment.

When we compute the modes with the EBC we obtain a qualitatively new result – modes with realistic decay time-scales. The modes are computed in the special case in which the buoyancy frequency is a constant $N = N_1$ in the troposphere, and has value N_2 different from N_1 at the bottom of the stratosphere. The temporal frequencies and speeds of the new modes are very close to those of the rigid lid, however, they have realistic decay time-scales of 1.5 days and one week for the first two baroclinic modes. High baroclinic modes have slower decay at a rate decreasing as $\sim n^{-2}$, where n is the vertical wave-number. In addition to that, we find a new "zero" baroclinic mode, which is stationary and has the fastest decay (time scale of about one hour), representing a fast adjustment mode. Note that the new decaying modes are computed for the case of a finite troposphere, yet they preserve the dissipative feature of the models with an infinite atmosphere.

One of the remarkable features of the new model is that it only involves one parameter α, which is a function of the ratio of the Brunt-Väilsälä frequencies N_1/N_2. By allowing N_1/N_2 to change between 0 and 1, we obtain a one-parameter family of models of the atmosphere. In particular, we show that the rigid lid approximation is correct when $N_1/N_2 \to 0$. In the limit $N_1/N_2 \to 1$ the boundary between the domains disappears and the EBC reduces to a radiation condition.

The paper is organized as follows. In section 1 we introduce the equations and the main assumptions, and compute the EBC in section 2. The new modes are introduced in section 3, and their features in section 4. Section 5 shows how to project the initial conditions onto the new modes.

1. Basic equations

We consider the following simple model of a semi-infinite atmosphere through linearized Boussinesq equations in hydrostatic balance:

$$\begin{aligned} \rho_0 u_t + p_x &= 0, \\ p_z + \rho &= 0, \\ \rho_t + w\frac{d\rho_0}{dz} &= 0, \\ u_x + w_z &= 0. \end{aligned} \tag{1}$$

Here x and z are the zonal and vertical coordinates, u and w are the horizontal and vertical components of the velocity, and p and ρ are the pressure and density perturbations from $p_0(z)$ and $\rho_0(z)$, which are in hydrostatic balance. For simplicity we consider here a 2D case. The extension to the non-rotating 3D case is shown in the Section 2. The equations have been non-dimensionalized using $\gamma = \tilde{H}/\tilde{L}$ as the long-wave parameter, where $\tilde{H}$ is the depth of the troposphere, and $\tilde{L}$ is the horizontal length scale. The scales for the buoyancy $\tilde{N}$, horizontal and vertical velocities $\tilde{u}$ and $\tilde{w}$ are as follows

$$\begin{aligned} \tilde{N} &= \sqrt{g/\tilde{H}}, \quad \tilde{t} = 1/(\gamma\tilde{N}) \\ \tilde{u} &= \tilde{L}/\tilde{t} = \sqrt{g\tilde{H}}, \quad \tilde{w} = \gamma\tilde{u}. \end{aligned} \tag{2}$$

Manipulating (1), one obtains an equation for the vertical velocity w alone:

$$\frac{(\rho_0 w_z)_{z\,tt}}{\rho_0} + w_{xx}N^2 = 0, \quad N^2 = -\frac{d\rho_0/dz}{\rho_0}.$$

After the change of variables

$$w = \phi/\sqrt{\rho_0},$$

the equation and its dispersion relation for constant N become

$$\begin{aligned} \phi_{zz\,tt} - \phi_{tt}\left(\frac{N^4}{4} + \frac{(N^2)'}{2}\right) + N^2\phi_{xx} = 0, \\ \omega^2 = \frac{N^2k^2}{m^2 + \frac{N^4}{4}}. \end{aligned} \tag{3}$$

Note: if N were uniform, we could absorb it into a slow time t; then the only place where high powers of N would appear is in the brackets below, which we effectively ignore (since N is small) for the rest of the article:

$$\begin{aligned} \phi_{zz\,tt} - \phi_{tt}\left(\frac{N^4}{4} + \frac{(N^2)'}{2}\right) + \phi_{xx} = 0, \\ \omega^2 = \frac{k^2}{m^2 + \left(\frac{N^4}{4}\right)}. \end{aligned} \tag{4}$$

2. Reflection and transmission coefficients and the effective boundary condition

To compute how much energy is transmitted from the troposphere to the stratosphere, we perform a local computation of reflection and transmission coefficients at the tropopause.

For the purposes of this section, the linearized boundary between the troposphere ($z < 0$) and the stratosphere ($z > 0$) is at $z = 0$. In a small neighborhood of the interface we consider N to be constant below and above the interface with the values N_1 and N_2. We introduce an upward-propagating plane wave in the troposphere and compute the reflection and transmission coefficients at the tropopause. The criterion to construct the EBS is that applying it at the boundary gives exactly the same reflection coefficient as if the stratosphere were there with $N = N_2$.

Consider an infinite domain with a linearized boundary $z = 0$, where the Brunt-Väisälä frequency N changes from N_1 ($z < 0$) to N_2 ($z > 0$). We introduce an incoming wave with an upward group velocity in the troposphere, which is partially reflected and transmitted. The (I) incoming, (R) reflected and (T) transmitted waves shown in (Fig. 1)

$$\begin{aligned}
\phi_I &= \quad \exp(i(k_I x + m_I z - \omega_I t)),\\
\phi_R &= R\exp(i(k_R x + m_R z - \omega_R t)),\\
\phi_T &= T\exp(i(k_T x + m_T z - \omega_T t)),
\end{aligned}$$

have the corresponding local dispersion relations

$$\begin{aligned}
\omega_I &= -\mathrm{sign}(m_I)\frac{N_1|k_I|}{\sqrt{m_I^2 + N_1^4/4}},\\
\omega_R &= \mathrm{sign}(m_R)\frac{N_1|k_R|}{\sqrt{m_R^2 + N_1^4/4}},\\
\omega_T &= -\mathrm{sign}(m_T)\frac{N_2|k_T|}{\sqrt{m_T^2 + N_2^4/4}}.
\end{aligned}$$

This takes into account that the group velocity is positive for the incoming and transmitted waves and negative for the reflected wave.

The boundary conditions at the interface are

$$\begin{aligned}
&[\phi] = 0, \quad [\phi'] = -\phi\frac{[N^2]}{2},\\
&\omega_I = \omega_R = \omega_T, \quad k_I = k_R = k_T = k, \qquad (5)
\end{aligned}$$

where the square brackets denote the jump across the interface. These equations represent continuity of the vertical velocity, horizontal velocity, frequency and horizontal wave number respectively. In terms of the plane wave parameters, the boundary conditions become

$$1 + R = T, \qquad [\phi'] + im_I(1 - R) = im_T T. \tag{6}$$

Assuming that $m_I^2 \gg \max\{N_1^2(N_2^2 - N_1^2)/4,\, N_1^4/4\}$, we obtain

$$m_T \approx \frac{N_2}{N_1} m_I\,, \quad \mathit{and} \quad \omega_I \approx -\frac{N_1|k|}{m_I}\,, \tag{7}$$

which allows us to manipulate (6) into

$$\begin{aligned} -\frac{1}{2}[N^2](1+R)i\omega_I + im_I(1-R)i\omega_I \\ = ik(1+R)(-i\,\mathrm{sign}(k))N_2. \end{aligned}$$

In terms of the original variables this gives us the effective boundary condition (EBC)

$$\frac{[N^2]}{2}\phi_t - \phi_{tz} \quad = \quad N_2 H(\phi_x), \tag{8}$$

where H is the Hilbert transform. Note that computing the Hilbert transform term does not add any numerical complications if one uses spectral methods for the horizontal coordinate.

The EBC can be easily generalized to the non-rotating 3D case, which includes the meridional direction y, as

$$\frac{[N^2]}{2}\phi_t - \phi_{tz} \quad = \quad N_2\sqrt{-\Delta}\,\phi.$$

Here $\sqrt{-\Delta}$, just as the operator $H\partial_x$ in 2D case, is a pseudo-differential operator; its Fourier transform is $\sqrt{k^2 + l^2}$, where l is the meridional wave-number.

3. The leaky rigid lid modes

We compute the leaky rigid lid modes for the case when N_1 is constant, and the jump of N across the boundary satisfies $N_2 - N_1 \neq 0$. We neglect the high order terms in N in (3), which leads to neglecting high order terms in N in the EBC (8). Thus, to obtain the new modes, we solve

$$\phi_{tt\,zz} = -N_1^2 \phi_{xx},$$

$$\phi_{\,tz} = -N_2 H(\phi_x) \quad \text{at} \quad z = 1,$$

$$\phi = 0 \quad \text{at} \quad z = 0.$$

Looking for the solution in the form $\phi(t, x, z) = e^{ikx} e^{\omega t} f(z)$ we find that

$$f(z) = \sinh\left(\frac{kN_1}{\omega} z\right).$$

The EBC at $z = 1$ then yields

$$\tanh\left(\frac{kN_1}{\omega}\right) = -\frac{N_1}{N_2}\text{sign}(k),$$

and therefore

$$\omega = -|k|N_1 / \tanh^{-1}(N_1/N_2).$$

We introduce a parameter $\alpha = \tanh^{-1}(N_1/N_2)$, which for a realistic atmosphere with $N_1/N_2 \approx 1/2$ has a value $\alpha \approx 1/2$. In terms of α,

$$\omega = -\frac{|k|N_1}{\alpha + i\,\pi n},$$

with n integer. Hence the decay rate and frequency of the modes are given respectively by

$$Re(\omega) = -\frac{|k|\,N_1\alpha}{\alpha^2 + (\pi n)^2} \sim -\frac{|k|N_1\alpha}{\pi^2 n^2} \quad \text{as} \quad n \to \infty, \tag{9}$$

$$Im(\omega) = \frac{|k|\,N_1 \pi n}{\alpha^2 + (\pi n)^2} \sim \frac{|k|\,N_1}{\pi n} \quad \text{as} \quad n \to \infty. \tag{10}$$

The corresponding vertical structure for the leaky rigid lid modes is given by

$$f(z) = -\sinh(\alpha z)\cos(\pi n z) + i\cosh(\alpha z)\sin(\pi n z). \tag{11}$$

4. Features of the leaky rigid lid modes

In this section we show that, for the realistic choices of N_1 and N_2, the first three leaky lid modes exhibit decay time-scales of 1 hour, 1.5 days and $\sim$ 1 week. All the modes converge to the rigid lid modes in the limit of $N_1/N_2 \to 0$. Finally, there exists an $n = 0$ mode, which is stationary and has the fastest decay time-scale of 1 hour.

a. Decay time-scales and slightly adjusted baroclinic speeds

The leaky rigid lid modes decay at the rate

$$Re(\omega) = -\frac{|k|\, N_1 \alpha}{\alpha^2 + (\pi n)^2} \sim -\frac{|k| N_1 \alpha}{\pi^2 n^2} \quad \text{as} \quad n \to \infty \tag{12}$$

For a realistic atmosphere with $\alpha = 1/2$, $\tilde{L} = 1000\,\text{km}$, and $\tilde{H} = 16\,\text{km}$, the decay time-scales are

$$T_n = \frac{\alpha^2 + (\pi n)^2}{\alpha |k| N_1} \frac{L}{\sqrt{gH}}$$

$$T_0 = 1\,\text{hr}, \quad T_1 = 1.3\,\text{days}, \quad T_2 = 5.2\,\text{days}, \tag{13}$$

and the speeds of the modes are respectively

$$v_0 = 0, \quad v_1 = 49\,\text{m/s}, \quad v_2 = 25\,\text{m/s}. \tag{14}$$

These are very close to the speeds of the rigid lid modes, which for the same values of the dimensional parameters are $51\,m/s$ and $25.4\,m/s$ for the first and the second baroclinic mode. Since the temporal frequency is

$$Im(\omega) = \frac{N_1 |k|}{\alpha^2/(\pi n) + \pi n},$$

for any α, as $n \to \infty$ the leaky lid frequencies and speeds approach those of the rigid lid modes. For the actual atmosphere this is true for all $n > 0$ because then $\alpha << \pi n$.

b. *Leaky modes as a correction to the classic rigid lid modes, and reappearance of the rigid lid in the limit of* $N_1/N_2 \to 0$

For realistic values of α the classic rigid lid mode $\sin(\pi n z)$ term shown in Fig. 4 is now only a part of the solution, with the slight modification in amplitude $\cosh(\alpha z)$, which is between 1 and 1.2 for $\alpha = 0.5$ (Fig. 3). The new part of the mode is the $\sinh(\alpha z)\cos(\pi n z)$ term (See Fig. 2), which is zero at the lower boundary $z = 0$, but always $\sinh(\alpha)$ at the top boundary $z = 1$. The full solution (corresponding to the modes $n = 0$, 1, 2 and 3), at time $t = 0$, and for $kx = 1$, is shown in Fig. 5.

One of the remarkable features of our model is the parameter $\alpha = \tanh^{-1}(N_1/N_2)$. The classic rigid lid approximation is only valid in the limit $\alpha \to 0$, which corresponds to $N_1/N_2 \to 0$. Indeed, the vertical structure function in this limit becomes $f(z) = \sin(\pi n z)$, the decay time-scales becomes infinite, and the leaky mode $n = 0$, described below, is no longer a non-trivial solution. The frequency of the modes converges to the rigid lid modes frequency $\omega = |k|N_1/\pi n$, and the speed of the modes becomes the speed of the baroclinic rigid lid modes $v = \text{sign}(k)N_1/\pi n$.

c. *Disappearing boundary as* $N_1/N_2 \to 1$

In the limit $N_1/N_2 \to 1$, which corresponds to $\alpha \to +\infty$, the boundary disappears. For example, for constant $N = N_1 = N_2$ the equation $\phi_{tt\,zz} = -N^2\phi_{xx}$ factors, yielding two independent wave solutions with upwards and downwards group velocities, of which the EBC selects only the upward-moving one:

$$\begin{aligned}
(\partial_{tz} - |k|N)(\partial_{tz} + |k|N)\phi &= 0, \\
(\partial_{tz} + |k|N)\phi &= 0, \quad \text{at} \quad z = 1 \\
\phi &= 0, \quad \text{at} \quad z = 0.
\end{aligned}$$

More generally, in this case discrete modes do not exist anymore.

d. The new $n = 0$ mode

We have discovered a new mode with $n = 0$, which is not present in the classical model. It has zero speed, decays on a time-scale of one hour, and does not have oscillations in the vertical. Its vertical structure for any value of α is simply

$$f(z) = \sinh(\alpha z).$$

In the rigid lid limit, the mode disappears as expected as $\alpha \to 0$.

This is a tropospheric mode with a barotropic flavor. Physically, it may represent the fast adjustment of the troposphere to global perturbations, such as those originating from deep convection.

5. Projecting onto the leaky rigid lid modes

Projecting the initial data or an external forcing onto the leaky modes is a non-trivial task, because the eigenvalue problem in the vertical is not standard, since it has an eigenvalue in one of the boundary conditions, and the corresponding leaky modes are not orthogonal. The approach that we have found the simplest is to map the original variable ϕ and its second derivative with respect to t and z onto new variables A and B, for which the projection reduces to a Fourier series decomposition, and then map them back into ϕ and ϕ_{tz}.

To do that we go back to the system (1) for ϕ, and look for solution in the form $\phi(x, z, t) = e^{ikx}\Phi(z, t)$. Then Φ satisfies

$$\Phi_{tt\,zz} = N_1^2 k^2 \Phi, \quad 0 < z < 1,$$

$$\Phi_{t\,z} = -\mu N_1 |k| \Phi, \quad \text{at} \quad z = 1,$$

$$\Phi = 0, \quad \text{at} \quad z = 0,$$

where $1/\mu = N_1/N_2 = \tanh^{-1}(\alpha)$. The values $\mu = 1$ and $\mu \to +\infty$ correspond to the vanishing boundary and the rigid lid respectively. Absorbing $N_1|k|$ into a new time τ, the equation simplifies:

$$
\begin{aligned}
&\Phi_{\tau\tau\, zz} = \Phi, \quad 0 < z < 1, \\
&\Phi_{\tau\, z} = -\mu\Phi, \quad \text{at} \quad z = 1, \\
&\Phi = 0, \quad \text{at} \quad z = 0.
\end{aligned} \tag{15}
$$

Motivated by the functional form of the leaky lid modes (11), we introduce new variables A and B related to Φ by the following transformation

$$
\begin{pmatrix} \Phi \\ \Phi_{z\,\tau} \end{pmatrix} = \begin{pmatrix} -\sinh(\alpha z) & \cosh(\alpha z) \\ \cosh(\alpha z) & -\sinh(\alpha z) \end{pmatrix} \begin{pmatrix} A \\ iB \end{pmatrix}. \tag{16}
$$

A straightforward calculation shows that under this transformation the equation (15) is equivalent to

$$
\begin{aligned}
&A_{\,z\tau} = iB + i\,\alpha B_\tau, \\
&B_{\,z\tau} = -iA - i\,\alpha A_\tau, \\
&B = 0 \quad at \quad z = 0, 1.
\end{aligned} \tag{17}
$$

Notice now that this last system is easily solvable using cosine Fourier series for A and sine Fourier series for B, where we can use the standard formulas to compute the coefficients. Furthermore, through the transformation from (A, B) to Φ it is easy to see that the above Fourier series become the leaky modes for Φ.

One may wonder how one arrives to the transformation above. That A and B should satisfy (17) follows from the form of the leaky modes, which suggests that A and B should be cosines and sines respectively. Hence, mode by mode one should have

$$
\begin{aligned}
A_z &= -(\pi n)B = -\left(\frac{1}{i\omega_n} + \frac{\alpha}{i}\right) B, \\
B_z &= (\pi n)A = \left(\frac{1}{i\omega_n} + \frac{\alpha}{i}\right) A.
\end{aligned}
$$

where we have used that $\omega_n = (-\alpha + i\pi n)^{-1}$. Therefore, multiplying these equations by ω_n and replacing $\omega_n B$ by B_τ and $\omega_n A$ by A_τ yields (17). The rest is obvious.

Conclusions

We offer a potential answer to the debate on whether the atmosphere should be modeled as infinite or finite, and whether there exist discrete modes at all: the troposphere can be studied in isolation, but with an effective boundary condition at the top which allows a fraction of the energy in the long waves to escape into the stratosphere. This approach is valid under the assumption that the waves that escape through the tropopause do not return after being reflected in stratospheric inhomogeneities. Our effective boundary condition gives the same reflection coefficient at the tropopause as if there were a stratosphere above with a prescribed buoyancy frequency. This self-contained model of the troposphere has the dissipative properties associated with the upward wave radiation of an infinite atmosphere, even though it has a discrete spectrum in the vertical. The new leaky rigid lid modes, which we compute assuming that the buoyancy frequency is piece-wise constant, decay in time with characteristic time-scales of 1 hour, 1.5 days and 1 week for the first three modes. These are typical relaxation times of many atmospheric phenomena, suggesting that upward wave radiation could be a key player in setting up these times, and providing a modeling scenario to study them.

In this article, we have concentrated on the new physics and mathematical formulation of the leaky rigid lid, for which we have adopted the simplest scenario of linear and irrotational waves. Further work is required to include the effects of vorticity and the Earth rotation. This would constitute an important step in the study of the interplay between wave radiation and eddies and storms.

6. Figures and tables

a. Figures

Acknowledgments.

The work of the authors was partially supported by grants from the National Science Foundation (NSF), as follows: L. G. Chumakova by NSF 0903008, R. R. Rosales by DMS 1007967 and DMS 0907955, and E. G. Tabak by DMS 0908077.

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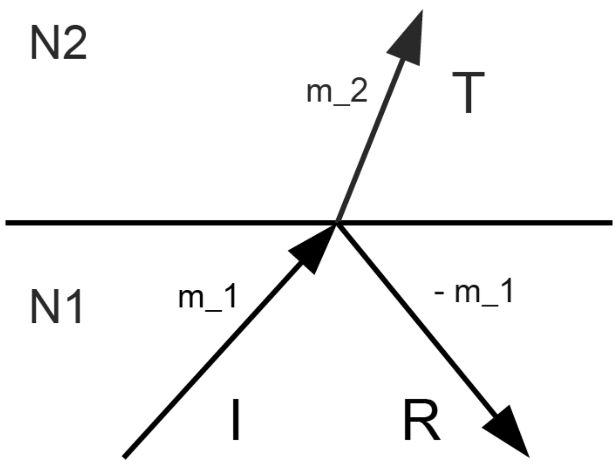

Fig. 1. Incoming, reflected and transmitted waves at the interface between the troposphere ($N = N_1$) and the stratosphere ($N = N_2$), where N is the Brunt-Väisälä frequency. The wave amplitudes are I, R and T and the vertical wave-numbers are m_1, $-m_1$ and m_2 respectively.

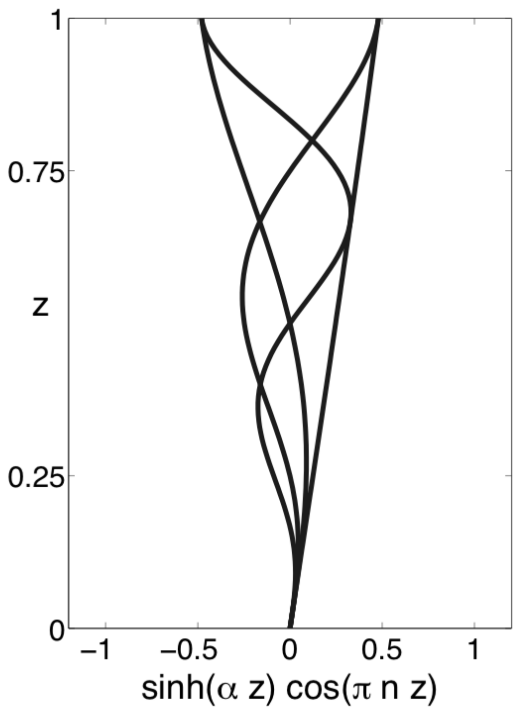

FIG. 2. Component $\sinh(\alpha z)\cos(\pi n z)$ ($n = 0$, 1, 2, 3) of the leaky lid modes generated by the dissipation, representing new behavior not present in the classic rigid lid modes.

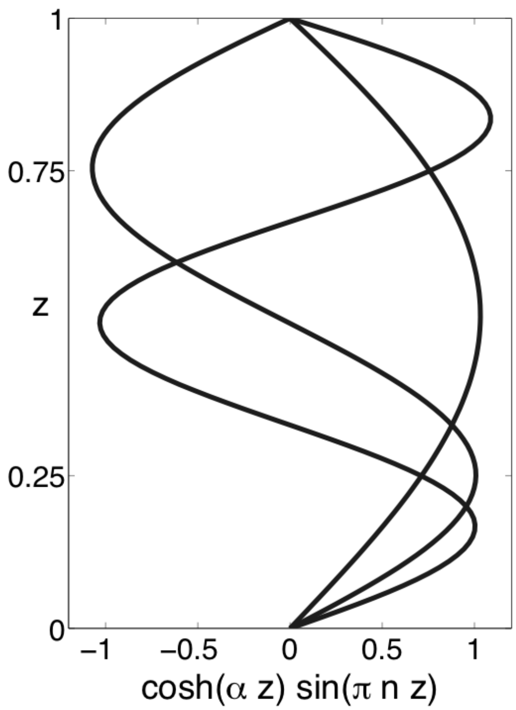

FIG. 3. Component $\cosh(\alpha z)\sin(\pi n z)$ ($n = 0,\ 1,\ 2,\ 3$) of the leaky lid modes, which corresponds to the classic rigid lid modes.

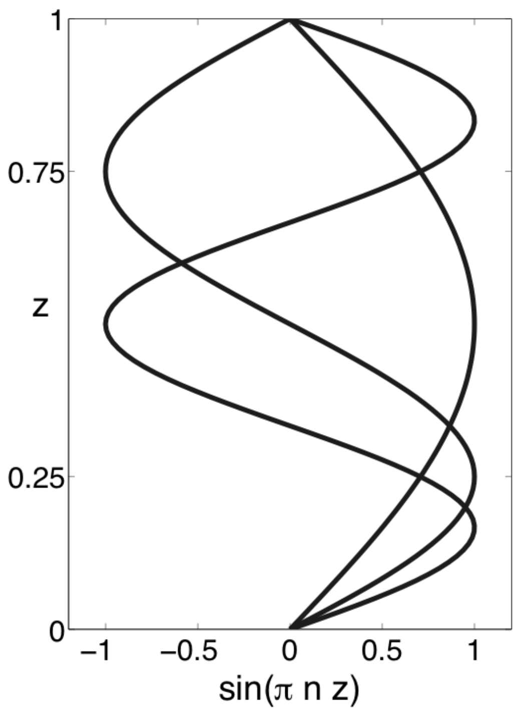

FIG. 4. The first three classic rigid lid modes.

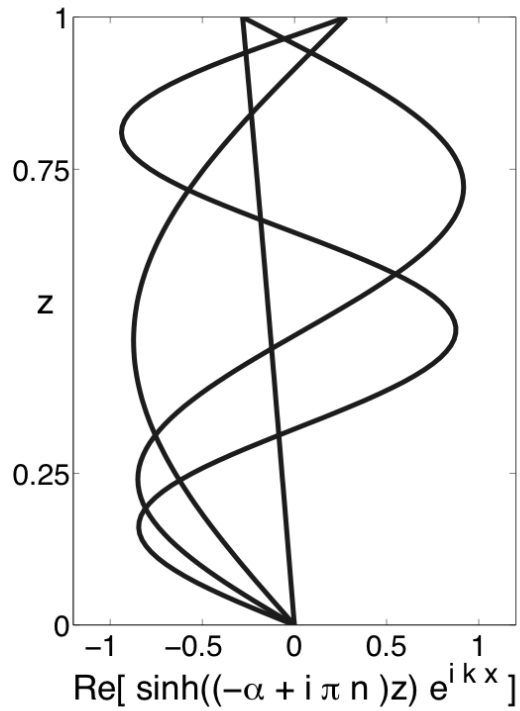

FIG. 5. The full solution corresponding to the first four leaky lid modes ($n = 0$, 1, 2, 3) at $t = 0$ and $kx = 1$.