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Area Between Curves.

Given two functions $f(x)$ and $g(x)$ with $f(x) > g(x)$ on $[a,b]$ the area betwen the curves is

$$\int_a^b [f(x) - g(x)]dx$$

Often the points a, b are not specified and we take it as the points where $f(x) = g(x)$ so that in $[a,b]$ one curve is above the other, with the curves meeting at a and b. We are talking of the area enclosed.

Example: Consider the parabola $y = x^2$ and the line $y = 2x + 1$.They intersect when $x^2 = 2x + 1$ or $x^2 - 2x - 1 = 0$. $x = 1 \pm \sqrt{2}$

$$Area = \int_{1-\sqrt{2}}^{1+\sqrt{2}} (2x + 1 - x^2)dx = [x^2 + x - \frac{x^3}{3}]|_{1-\sqrt{2}}^{1+\sqrt{2}} = \frac{13}{3}\sqrt{2}$$

Example: Consider the area between $f(x) = 1 - x^2$ and $y = 0$. This is just

$$\int_{-1}^{!} (1 - x^2)dx = 2 - \frac{2}{3} = \frac{4}{3}$$

Alternately we can view $x = \pm\sqrt{1-y}$ as the curves and calculate the area by horizontal strips. The length of a horizontal line (infinitesimal strip) through y is $2\sqrt{1-y}$ and y can go from 0 to 1.

$$Area = \int_0^1 2\sqrt{1-y}\ dy = -2\frac{(1-y)^{\frac{3}{2}}}{\frac{3}{2}}|_0^1 = \frac{4}{3}$$

The main steps are, pick a direction, usually x or y coordinate axis. Find, as a function of the point on this axis, the length of the line perpendicular to it that is inside the region whose area is to be calculated. Then integrate this over the appropriate range.

Volumes.

Volumes are calculated by again picking a direction, (i.e a line) and cutting the region by planes perpendicular to it, calculating the rear as function of the point on the line and integrating this area over the appropriate range.

Example: Volume of a sphere. If the sphere is of radius r and we take the vertical axis from the South pole to north pole, the location on this axis can vary from $-r$ to $+r$. If we cut horizontally at location t, we get a circle. The radius of the circle is

$$f(t) = \sqrt{r^2 - t^2}$$

with an area of $\pi f(t)^2$ and the volume is

$$\pi \int_{-r}^{r} (r^2 - t^2) dt = \pi(2r^3 - \frac{2}{3}r^3) = \frac{4}{3}\pi r^3$$

Example: Volume of a cone. Base radius is r and the height is h. At height y the horizontal cross section is a circle with radiurs $r(t)$ determined by similar triangles:

$$\frac{r(t)}{r} = \frac{h-t}{h}$$

$h - t$ being the height of the small cone above the cutoff. Area of cross section is

$$A(t) = \pi \frac{r^2}{h^2}(h-t)^2$$

and

$$\int_0^h A(t) dt = \frac{1}{3}\pi r^2 h$$

Volumes of revolution.

Certain bodies can be thought of as an arae revolving around an axis. For exaple if the semicircle $x^2 + y^2 \leq 1, x \geq 0$ is spun around the y axis we get the sphere of radius r. If a triangle $x \geq 0, y \geq 0, ax + by \leq 1$, with $a, b > 0$ is spun around the y axis we get a cone. If a circle of radius r, with center at $(a, 0)$ with $a > r$ is spun around the y axis we get a donut.Volumes of these objects can be calculated by t a common method.

A thin cylindrical shell of radius r and height h will have a volume roughly equal to the surface area times the thickness. The volume from this shell will be $2\pi rhdr$ where dr is the thickness of the shell. The body consists of layers of concentric shells the radius r which is also the distance from the axis of revolution, ranging over $r_1 \leq r \leq r_2$. The height a function $h(r)$ of the radius. The volume is given by

$$V = \int_{r_1}^{r_2} 2\pi r \, h(r) \, dr$$

Example: Consider the sphere which is the body obtained by revolving a semicircle of radius a around its diameter. The distance r from the diameter goes from 0 to a. The height is $2\sqrt{a^2 - r^2}$ which is the length of the chord at distance r. The volume therefore is

$$V = \int_0 *a 2\pi \cdot r \cdot 2\sqrt{a^2 - r^2} dr = 4\pi a^3 \int_0^1 y\sqrt{1 - y^2} dy = -4\pi a^3 \cdot \frac{1}{3} \cdot [1 - y^2]^{\frac{3}{2}} \Big|_0^1 = 4\pi a^3 \cdot \frac{1}{3}$$

Example: The cone of base radius r and height h is obtained by revolving the triangle in the first quadrant below the line $\frac{x}{r} + \frac{y}{h} = 1$ about the y axis. At distance x the height is $h(1 - \frac{x}{r})$. The volume is then

$$V = 2\pi h \int_0^r x(1 - \frac{x}{r})dx = 2\pi h r^2 \int_0^1 x(1 - x)\,dx = \frac{1}{3}\pi r^2 h$$

Example: Donut. A circle $(x - a)^2 + y^2 = r^2$ with $a > r$ revolves around the y axis to create a donut. At a distance x that runs from $a-r$ to $a+r$, the height is $2\sqrt{r^2 - (a - x)^2}$. The radius is of course x. The volume of the donut is

$$\begin{aligned} V &= 2\pi \int_{a-r}^{a+r} x \cdot 2\sqrt{r^2 - (a - x)^2}\,dx = 4\pi \int_{-r}^{r} (a + y)\sqrt{r^2 - y^2}\,dx \\ &= 4\pi r^2 \int_{-1}^{1} (a + ry)\sqrt{1 - y^2}\,dy = 4\pi r^2\, a \int_{-1}^{1} \sqrt{1 - y^2}\,dy \\ &= 4\pi r^2 a \cdot \frac{\pi}{2} = 2\pi^2 r^2 a \end{aligned}$$

Example: We can also slice the donut horizontally at heights x varying from $-r$ to r. Each cross section will be an annulus. Outer radius will be $a + \sqrt{r^2 - x^2}$ and the inner radius $a - \sqrt{r^2 - x^2}$. Area of annulus is

$$\pi[(a + \sqrt{r^2 - x^2})^2 - (a - \sqrt{r^2 - x^2})^2] = 4\pi a\sqrt{r^2 - x^2}$$

Volume is given by

$$4\pi a \int_{-r}^{r} \sqrt{r^2 - x^2}dr = 4\pi\, r^2\, a \int_{-1}^{1} \sqrt{1 - x^2}\,dx = 4\pi r^2 a \cdot \frac{\pi}{2} = 2\pi^2 r^2 a$$

Homework.

Q1. Find the area in the first quadrant between the parabola $x = y^2 + 1$ and the line $y = 2(x - 1)$

Q2. What is the volume of the solid generated when this region is rotated around the y axis?

Q3. What is the volume of the solid generated when this region is rotated around the x axis?

Q4, The region which is the triangle with the three corners at $\{(1, 0), (2, 0), (2, 3)\}$ is rotated around the y axis. Find the volume of the solid that is generated.

Q5, What do we get when the region is rotated around the x axis and what is the volume of that solid?