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Arc length.

If we consider the graph of $y = f(x)$ as a curve, the area under the curve between $x = a$ and $x = b$ is calculated by the integral

$$Area = \int_a^b f(x)dx$$

But the length of the curve has to be computed differently. Because the length also adds up as we go along the curve it also will be given by an integral. The are a calculation depended on the fact that a small infinitesimal are was $f(x)\Delta x$. But infinitesimal arc length is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + (\frac{\Delta y}{\Delta x})^2}\ \Delta x = \sqrt{1 + [f'(x)]^2}\Delta x$$

Adding them up leads to

$$Length = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example. Arc length of a quarter circle. $0 \leq x \leq r$

$$f(x) = \sqrt{r^2 - x^2}$$

$$1 + [f'(x)]^2 = 1 + \frac{x^2}{r^2 - x^2}$$

$$\int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = \int_0^r \sqrt{\frac{r^2}{r^2 - x^2}}\, dx = r \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi r}{2}$$

Example. Length of the parabolic arc $y = x^2$ from $x = 0$ to $x = 3$.

$$1 + [f'(x)]^2 = 1 + 4x^2$$

and

$$Length = \int_0^1 \sqrt{1 + 4x^2}\, dx$$

Substitute $x = 2(t - \frac{1}{t})$. $dx = 2(1 + \frac{1}{t^2})$ $x = 0, t = 1$. $x = 3, t = 2$.

$$Length = 4\int_1^2 (t + \frac{1}{t})(1 + \frac{1}{t^2})dt = 4\int_1^2 [t + \frac{2}{t} + \frac{1}{t^3}]dt = 7 + 8\log 2$$

We can think of the arc as defined by $x = \sqrt{y}$ from $y = 0$ to $y = 9$.

$$Length = \int_0^9 \sqrt{1 + \frac{1}{4y}} dy$$

which will reduce to the other integral by changing variable $y = x^2$.

Parametric representation of curves. A curve can be described by the motion of a person traversing it. $x = x(t), y = y(t)$ as t varies from a to b describes a curve going from $(x(a), y(a))$ to $(x(b), y(b))$. The length is

$$\sum \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sum \sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2} \Delta t \to \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

Example. A full circle is described by $x = \cos t, y = \sin t$, $0 \leq t \leq 2\pi$.

$$[x'(t)]^2 + [y'(t)]^2 = 1$$

$$length = \int_0^{2\pi} 1\, dt = 2\pi$$

Example. The length of the spiral $x(t) = e^{-t} \cos t, y(t) = e^{-t} \sin t$ between $t = 0$ and $t = 1$ can be calculated as

$$\begin{aligned}
&\int_0^1 \sqrt{(-e^{-t} \cos t - e^{-t} \sin t)^2 + (-e^{-t} \sin t + e^{-t} \cos t)^2} dt \\
&= \int_0^1 e^{-t} \sqrt{(\sin t + \cos t)^2 + (\sin t - \cos t)^2} dt \\
&= \sqrt{2} \int_0^1 e^{-t} dt = \sqrt{2}[1 - e^{-1}]
\end{aligned}$$

The full spiral tends to the origin and the total length is finite and equals $\sqrt{2}$.

Example. The Cycloid. $x(t) = a(t - \sin t)$, $y = a(1 - \cos t)$.

$$[x'(t)]^2 + [y'(t)]^2 = a^2[(1 - \cos t)^2 + (\sin t)^2 = a^2[2(1 - \cos t)] = [2a \cos \frac{t}{2}]^2$$

Length from $t = 0$ to $t = 2\pi$ is

$$2a \int_0^{2\pi} |\cos \frac{t}{2}| dt = 4a \int_0^{\pi} |\cos x| dx = 8a$$

If $a = 1$, the "diameter" is 2π and half the "circumference" is 8.

Homework.

Q1.A top portion from the sphere of radius 6 is cut off at a horizontal plane 3 units from the top. i.e 3 units above the center. What is its volume?

Q2. The region below the curve $y = x^2(x-1)^2$ between $x = 0$ and $x = 1$ is rotated around the x-axis. What is the volume of the solid generated.

Q3. The region below the curve $y = x^2(x-1)^2$ between $x = 0$ and $x = 1$ is rotated around the y-axis. What is the volume of the solid generated.

Q4. An ellipse has parametric representation $x = a\cos t$, $y = b\sin t$ $0 \le t \le 2\pi$. Can you write a formula for its total length? Do not waste your time trying to calculate it.

Q5. Find the length of the curve $y = \log \frac{e^x+1}{e^x-1}$ between $x = a$ and $x = b$. $b > a > 0$.