One way to construct a diffusion process corresponding to the operator

$$(Lf)(x) = \frac{1}{2}\sum_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_j b_j(x)\frac{\partial u}{\partial x_j}(x)$$

is to find a process with the property

$$x(t+h) - x(t) \simeq \sqrt{h}Z + hb(x(t))$$

where Z is a Gaussian with dispersion $a_{i,j}(x(t))$. If $\sigma(x)\sigma^*(x) = a(x)$, $\sigma(x)$ maps $R^k \to R^d$ then $Z = \sigma(x)[\beta(t+h) - \beta(t)]$ should work. This leads to

$$dx(t) = \sigma(x(t)) \cdot d\beta(t) + b(x(t))dt$$

There are three possible equivalent formulations of what a solution to the equation means.

For any $x \in R^d$, there is a measure P on $\Omega = C[[0,T]; R^d$ such that $P[x(0) = 0] = 1$, and for any smooth f with compact support on R^d

$$f(x(t)) - f(x) - \int_0^t (Lf)(x(s))ds$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$.

Or equivalently there is a measure space and a filtration $(\Omega, \mathcal{F}_t, P)$ and two progressively measurable almost surely continuous processes $x(t,\omega), \beta(t,\omega)$ with values in R^d and R^k respectively, where $\beta(t,\omega)$ is a k-dimensional Brownian motion adapted to $\mathcal{F}_t$, i.e. for any $t > s$, $\beta(t) - \beta(s)$ is independent of $\mathcal{F}_s$. They satisfy

$$x(t) = x(0) + \int_0^t \sigma(x(s)) \cdot d\beta(s) + \int_0^t b(x(s))ds$$

This can be rephrased as finding a measure Q on $(C[[0,T], R^d \times R^k), \mathcal{F}_t, Q)$ such that for smooth f with compact support on $R^d \times R^k$

$$f(x(t), y(t)) - f(x(0), 0) - \int_0^t (Lf)(x(s), y(s))ds$$

is a martingale, where

$$(Lf)(x,y) = \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x,y) + \sum_{j=1}^d b_j(x)\frac{\partial f}{\partial x_j}(x,y)$$
$$+ \frac{1}{2}\sum_{i=1}^k \frac{\partial^2 f}{\partial^2 y_i}(x,y) + \sum_{i=1}^d\sum_{j=1}^k \sigma_{i,j}(x,y)\frac{\partial^2 f}{\partial x_i \partial y_j}(x,y)$$

Or one can ask if on the canonical Brownian motion space $(C[[0,T], R^k), \mathcal{F}_t, P)$ the equation

$$x(t) = x(0) + \int_0^t \sigma(x(s)) \cdot d\omega(s) + \int_0^t b(x(s))ds$$

can be solved with a progressively measurable almost surely continuous solution $x(t, \omega)$.

If σ satisfies the Lipschitz condition $\|\sigma(x) - \sigma(y)\| \leq C|x - y|$, then for any initial random variable ξ measirable w.r.t. $\mathcal{F}_0$ with $\|\xi_2\|_2 < \infty$ the above equation has a unique solution.

Existence. Let us define recursively starting with $x_0(t) \equiv \xi$

$$x_{n+1}(t) = \xi + \int_0^t \sigma(x_n(s)) \cdot d\beta(s) + \int_0^t b(x_n(s))ds$$

Inductively σ_n is progressively measurable and bounded. Hence so is $x_n(t)$. Taking the difference

$$x_{n+1}(t) - x_n(t) = \int_0^t [\sigma(x_n(s)) - \sigma(x_{n-1}(s))] \cdot d\beta(s) + \int_0^t [b(x_n(s)) - b(x_{n-1}(s))]ds$$

Let us denote by $\Delta_n(t) = E[\sup_{0 \leq s \leq t} \|x_{n+1}(s) - x_n(s)\|^2]$. Then

$$\Delta_n(t) \leq 2E[\sup_{0 \leq s \leq t} |\int_0^s [\sigma(x_n(\tau)) - \sigma(x_{n-1}(\tau))] \cdot d\beta(\tau)|^2 + |\int_0^t |b(x_n(s)) - b(x_{n-1}(s))|ds|^2]$$

By Doob's inequality the first term is dominated by $8E[\int_0^t |\sigma(x_n(\tau)) - \sigma(x_{n-1}(\tau))|^2]$ and the second by $2TE[| \int_0^t |b(x_n(s)) - b(x_{n-1}(s))|^2 ds]$. If we consider a finite interval $[0, T]$. using the Lipschitz condition

$$\Delta_{n+1}(t) \leq C(T) \int_0^t \Delta_n(s)ds$$

with

$$\Delta_0(t) = 8E[\|\sigma(\xi) \cdot (\beta(t) - \beta(0))\|^2 + 2Tt\|b(\xi)\|^2] \leq c(T)t$$

By induction

$$\Delta_n(t) \leq \frac{[C(T)]^{n+1}}{(n+1)!}$$

Since $\sum_n \sqrt{\Delta_n(t)} < \infty$, it follows that

$$P[\sum_n \sup_{0 \leq t \leq T} \|x_{n+1}(t) - x_n(t)\| < \infty] = 1$$

Therefore $\lim_{n\to\infty} x_n(t) = x(t)$ exists almost surely and passing to the limit

$$x(t) = x(0) + \int_0^t \sigma(x(s)) \cdot d\omega(s) + \int_0^t b(x(s))ds$$

Uniqueness. For $i = 1, 2$

$$x_i(t) = \xi + \int_0^t \sigma(x_i(s)) \cdot d\beta(s) + \int_0^t b(x_i(s))ds$$

Let $y(t) = x_1(t) - x_2(t)$ and $\delta(t) = E[\|y(t)\|^2]$.

$$\delta(t) \le C(T) \int_0^t \delta(s)ds$$

Implies $\delta(t) \equiv 0$.

Markov and Strong Markov Property.

If you start the solution from $x(0) = x$ and run it up to a stopping time τ, then the solution starting from $x(\tau)$ is the same as the old one. But the Brownian increments after time τ are independent of $\mathcal{F}_\tau$. This is strong Markov property. The discrete analog is if $X_{n+1} = f(X_n, Y_{n+1})$ where $\{Y_n\}$ are mutually independent and indpendent of X_0, then $\{X_n\}$ is a Markov process.

If the SDE

$$x(t) = x + \int_0^t \sigma(x(s)) \cdot d\beta(s) + \int_0^t b(x(s))ds$$

has a unique solution for some choice of σ satisfying $\sigma(x)\sigma^*(x) = a(x)$ then the Markov family $\{P_x\}$ the distributions of $(\cdot)$ for the varying starting points $x \in R^d$, are solutions to the martingale problem for L. Does it imply that there are no other solutions to the Martingale Problem?

Theorem.. If P is any solution of the martingale problem and if $a(x) = \sigma(x)\sigma^*(x)$ for some choice of σ for which the solution to the SDE is unique then the soulution to the martingale problem is unique.

Proof. If P_1 and P_2 are two solutions to the martingale probem and $a(x) = \sigma(x)\sigma^*(x)$ then on the space $\Omega = C[[0, T]; R^d \times R^k]$ there are two probability measures $\hat{P}_1$ and $\hat{P}_2$. There projections on $C[[0, T]; R^d]$ are P_1 and P_2 whlie their projections on the second component $C[[0, T]; R^k]$ is the Brownian motion μ. They are related by

$$x(t) = x + \int_0^t \sigma(x(s)) \cdot d\beta(s) + \int_0^t b(x(s))ds$$

with $\omega \in \Omega$ being $\omega(t) = (x(t), \beta(t))$ If we can construct a measure $\hat{Q}$ on $\hat{\Omega} = C[[0, T]; R^d \times R^d \times R^k]$ with $\hat{\omega} = (x(t), y(t), \beta(t))$ such that $(x(\cdot), \beta(\cdot))$ and $(y(\cdot), \beta(\cdot))$ have distributions $\hat{P}_1$ and $\hat{P}_2$ then we would have

$$x(t) = x + \int_0^t \sigma(x(s)) \cdot d\beta(s) + \int_0^t b(x(s))ds$$

as well as

$$y(t) = x + \int_0^t \sigma(y(s)) \cdot d\beta(s) + \int_0^t b(y(s))ds$$

implying $x(t) \equiv y(t)$ proving $P_1 = P_2$.

Construction of $\hat{Q}$. Let us denote by $q_\beta^1(d\omega_1)$ the regular conditional probability distribution of $x(\cdot)$ given $\beta(\cdot)$ and by $q_\beta^2(d\omega_1)$ the regular conditional probability distribution of $y(\cdot)$ given $\beta(\cdot)$. We define $\hat{Q}$ by

$$Q(d\omega_1, d\omega_2, d\beta) = q_\beta^1(d\omega_1) \times q_\beta^2(d\omega_1) \times \mu(d\beta)$$

We need to make sure that the Brownian increments after time t are independent of the σ-field $\mathcal{F}_t$ generated by $\{x(s), y(s), \beta(s)\}$ where $0 \le s \le t$. This is easily ckecked. Depends on the fact that the conditional distribution of $x(\cdot), y(\cdot)$ on $[0, t]$ given $\beta(\cdot)$ on $[0, T]$ depends only on $\beta(\cdot)$ on $[0, t]$.

Given $a(x)$ when can we find a $\sigma(x)$ with $\sigma(x)\sigma^*(x) = a(x)$?