Lemma. If $a(x)$ is Lipschitz and $a(x) \geq c > 0$, then $\sigma(x) = \sqrt{a(x)}$ is Lipschitz. If $a(x) \geq 0$ and has a bounded second derivative then $\sigma(x) = \sqrt{a(x)}$ is again Lipschitz. The same is true for positive semidefinite matrices. For $\{a_{i,j}(x)\}$ to have a square root $\sigma(x) = \sqrt{a(x)}$ which is Lipschitz either $a(x)$ to be Lipschitz and bounded below $a(x) \geq cI$ for some $c > 0$ or $a_{i,j}(x)$ to have two bounded derivatives is sufficient.

Proof. If $a(x) \geq c > 0$

$$|\frac{d\sqrt{a(x)}}{dx}| = \frac{1}{2\sqrt{a(x)}}|\frac{da(x)}{dx}| \leq \frac{1}{2\sqrt{c}}|\frac{da(x)}{dx}|$$

On the other hand if $0 \leq a(x) \leq c_1$ and $|a''(x)| \leq c_2$, then by Taylor's theorem

$$0 \leq a(x+z) = a(x) + za'(x) + a''(\xi)\frac{1}{2}z^2 \leq a(x) + za'(x) + \frac{c_2}{2}z^2$$

This implies

$$[a'(x)]^2 \leq 2c_2 a(x) \leq 2c_1 c_2$$

In the higher dimensional case we have a symmetric positive semidefinite σ satisfying $\sigma^2(x) = a(x)$ or $\sum_k \sigma_{i,k}(x)\sigma_{k,j}(x) = a_{i,j}(x)$. We can diagonalize $a(0)$ and $\sigma(0)$ simultaneously. Taking a directional derivative D at 0

$$\sum_k [D\sigma_{i,k}](0)\sigma_{k,j}(0) + \sum_k \sigma_{i,k}(0)[D\sigma_{k,j}](0) = [Da_{i,j}](0)$$

or

$$[D\sigma_{i,j}](0)[\sigma_{j,j}(0) + \sigma_{i,i}(0)] = [Da_{i,j}](0)$$

$$|(D\sigma_{i,j})(0)| = \frac{|(Da_{i,j})(0)|}{[\sigma_{j,j}(0) + \sigma_{i,i}(0)]} = \frac{|(Da_{i,j})(0)|}{\sqrt{a_{j,j}(0)} + \sqrt{a_{i,i}(0)}} \leq \frac{|(Da_{i,j})(0)|}{2\sqrt{\lambda}} \leq \frac{\|a\|_1}{2\sqrt{\lambda}}$$

We can take sup over all directions D and take linear combinations $S^*\sigma S$ of σ, where S was used to diagonalize a. $\lambda = \inf_x \inf_{\|u\|=1} \langle a(x)u, u\rangle$.

If $a(x)$ is onlyi positive semidefnite since $a_{i,i} \pm 2a_{i,j} + a_{j,j} \geq 0$ it follows that if c is a bound on the second derivatives

$$|D[a_{i,i} \pm 2a_{i,j} + a_{j,j}]| \leq \sqrt{2c}\sqrt{a_{i,i} \pm 2a_{i,j} + a_{j,j}} \leq C\sqrt{a_{i,i} + a_{j,j}} \leq C[\sqrt{a_{i,i}} + \sqrt{a_{j,j}}]$$

Taking the difference of the two

$$|Da_{i,j}(x)| \leq C[\sqrt{a_{i,i}(x)} + \sqrt{a_{j,j}(x)}]$$

providing a bound for $|D\sigma_{i,j}(x)|$

Theorem (Surgery). Let L^1, L^2 be two operators

$$(L^{\{k\}}f)(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}^{\{k\}}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i^{\{k\}}(x)\frac{\partial f}{\partial x_i}(x)$$

Let $P^{\{1\}}$ be a solution to the martingale problem on $C[[0,T];R^d]$ corresponding to $L^{\{1\}}$ with $P[x(0) = x_0] = 1$. Let $\{P_x^{\{2\}}\}$ be a family of solutions corresponding to $L^{\{2\}}$ with $P_x^{\{2\}}[x(0) = x] = 1$. Let U be a neighborhood of x_0 such that for $x \in U$, $a_{i,j}^{\{1\}}(x) = a_{i,j}^{\{2\}}(x)$ for all i,j and $b_i^{\{1\}}(x) = b_i^{\{2\}}(x)$ for all i. For any path $x(\cdot)$ let $\tau = \inf\{t : x(t) \notin U\}$. Let us define a measure Q on $C[[0,T];R^d]$ by $Q = P^{\{1\}}$ on $\mathcal{F}_{\tau\wedge T}$ and r.c.p.d of $Q|\mathcal{F}_{\tau\wedge T} = P_{x(\tau\wedge T)}^{\{2\}}$ for $\tau \le t \le T$ on the set $\tau < T$. Then Q is a solution for $L^{\{2\}}$ with $Q[x(0) = x_0]$.

Proof depends on a simple lemma.

Lemma. Let $(\Omega, \mathcal{F}_t, \mathcal{F}, P)$ be a measure space with a filtration. Let τ be a stopping time. Let Q_ω^τ be the r.c.p,.d. of $P|\mathcal{F}_\tau$. That is to say that for almost all ω w.r.t. P¡

$$Q_\omega^\tau(A) = \mathbf{1}_A(\omega)$$

for all $A \in \mathcal{F}_\tau$ and $B \in \mathcal{F}$,

$$P(A \cap B) = \int_A Q_\omega^\tau(B)dP$$

Let $X(t)$ be a martingale with respect to $(\Omega, \mathcal{F}_t, P)$. Then for almost all ω w.r.t P, $\{X(t) : t \ge \tau(\omega)\}$ is a martingale respect to $(\Omega, \mathcal{F}_t, Q_\omega^\tau)$. Conversely if $x(t)$ is a martingale w.r.t $(\Omega, \mathcal{F}_t, Q_\omega^\tau)$ for times $t \ge \tau(\omega)$ for almost all ω with respect to P and $X(\tau \wedge t)$ is a martingale w.r.t $(\Omega, \mathcal{F}_t, P)$, then $X(t)$ is a martingale w.r.t. $(\Omega, \mathcal{F}_t, P)$.

We will defer the proof of the lemma but prove the thorem assuming the lemma.

Proof of Throrem. We need to prove that for any smooth function with compact support

$$X(t) = f(x(t)) - f(x(0)) - \int_0^t (L^{\{2\}}f)(x(s))ds$$

is martingale with respect to $(\omega, \mathcal{F}_t, Q)$. Let $\tau = \inf\{t : x(t) \notin U\}$. It is enough to show that $X(\tau \wedge t)$ is a martingale and $X(t)$ for $t \ge \tau$ is a martingale w.r.t Q_ω^τ the r.c.p.d of $Q|\mathcal{F}_\tau$. Since $Q_\omega^\tau = P_{x(\tau\wedge T)}^{\{2\}}$ and $X(t) = Y(t)$ until time τ where

$$Y(t) = f(x(t)) - f(x(0)) - \int_0^t (L^{\{1\}}f)(x(s))ds$$

it follows that $X(t)$ is martingale with respect to $(\omega, \mathcal{F}_t, Q)$

Proof of lemma. What does "for almost all ω w.r.t P, $\{X(t) : t \ge \tau(\omega)\}$ is a martingale respect to $(\Omega, \mathcal{F}_t, Q_\omega^\tau)$" mean?. For $A \in \mathcal{F}_{t_1}$,

$$\int_{A\cap\{\tau\le t_1\}} X(t_2)dQ_\omega^\tau = \int_{A\cap\{\tau\le t_1\}} X(t_1)dQ_\omega^\tau$$

for almost all ω with respect to P (modulo problems with null sets!). This requires for $B \in \mathcal{F}_\tau$

$$\int_B dP \int_{A\cap\{\tau\le t_1\}} X(t_2)dQ^\tau_\omega = \int_B dP \int_{A\cap\{\tau\le t_1\}} X(t_1)dQ^\tau_\omega$$

Since $A \cap \{\tau \le t_1\}$ is $\mathcal{F}_\tau$ measurable this reduces to verifying

$$\int_{A\cap B\cap\{\tau\le t_1\}} X(t_2)dP = \int_{A\cap B\cap\{\tau\le t_1\}} X(t_1)dP$$

Since $A, B \in \mathcal{F}_{t_1}$ it is true.

For the converse if $A \in \mathcal{F}_{t_1}$

$$\int_A X(t_2)dP = \int_{A\cap\{\tau\le t_1\}} X(t_2)dP + \int_{A\cap\{t_2\ge\tau>t_1\}} X(t_2)dP + \int_{A\cap\{\tau>t_2\ge\tau\}} X(t_2)dP$$

$$\begin{aligned}\int_{A\cap\{\tau\le t_1\}} X(t_2,\omega)P(d\omega) &= \int_A P(d\omega)\int_{\tau\le t_1} X(t_2,\omega')Q^\tau_\omega(d\omega')\\ &= \int_A P(d\omega)\int_{\tau\le t_1} X(t_1,\omega')Q^\tau_\omega(d\omega')\\ &= \int_{A\cap\{\tau\le t_1\}} X(t_1,\omega)P(d\omega)\end{aligned}$$

$$\begin{aligned}\int_{A\cap\{t_1<\tau\le t_2\}} X(t_2,\omega)P(d\omega) &= \int_{A\cap\{t_1<\tau\}} P(d\omega)\int_{\tau\le t_2} X(t_2,\omega')Q^\tau_\omega(d\omega')\\ &= \int_{A\cap\{t_1<\tau\le t_2\}} X(\tau,\omega)P(d\omega)\\ &= \int_{A\cap\{t_1<\tau\le t_2\}} X(\tau\wedge t_2,\omega)P(d\omega)\end{aligned}$$

$$\int_{A\cap\{\tau>t_2\}} X(t_2,\omega)P(d\omega) = \int_{A\cap\{\tau>t_2\}} X(\tau\wedge t_2,\omega)P(d\omega)$$

Adding the two

$$\begin{aligned}\int_{A\cap\{t_1<\tau\}} X(t_2,\omega)P(d\omega) &= \int_{A\cap\{t_1<\tau\}} X(\tau\wedge t_2,\omega)P(d\omega)\\ &= \int_{A\cap\{t_1<\tau\}} X(\tau\wedge t_1,\omega)P(d\omega)\\ &= \int_{A\cap\{t_1<\tau\}} X(t_1,\omega)P(d\omega)\end{aligned}$$

Since

$$\int_{A\cap\{\tau\le t_1\}} X(t_2,\omega)P(d\omega) = \int_{A\cap\{\tau\le t_1\}} X(t_1,\omega)P(d\omega)$$

adding the two

$$\int_A X(t_2,\omega)P(d\omega) = \int_A X(t_1,\omega)P(d\omega)$$

Uniqueness. (Localization). Suppose

$$L^{\{\alpha\}} = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}^{\{\alpha\}}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i^{\{\alpha\}}(x)\frac{\partial}{\partial x_i}$$

is a family of operators for which the solution to the martingale problem is unique, i.e for each α and x i are s a unique solution $\{P_x^{\{\alpha\}}\}$ such that $\{P_x^{\{\alpha\}}\}[x(0) = x] = 1$ and for any smooth f with compact support

$$f(x(t)) - f(x(0)) - \int_0^t (L^{\{\alpha\}}f)(x(s))ds$$

is a martingale w.r.t. $(C[[0,T],R^d],\mathcal{F}_t,P_x^{\{\alpha\}})$. Let

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

be such that for every $x \in R^d$ there is an $\epsilon(x)$ and $\alpha(x)$ such that $L^{\{\alpha(x)\}} = L$ on $B(x,\epsilon(x))$. We can assume with out loss of generality that $\epsilon(x)$ has a uniform lower bound on compact sets $|x| \le \ell$ for every ℓ. Then for any $x \in R^d$ there is at most one solution P_x such that $P[x(0) = x]$ and

$$f(x(t)) - f(x(0)) - \int_0^t (Lf)(x(s))ds$$

is a martingale for any smooth f with compact support.

Proof. Let if possible P_1 and P_2 be two solutions for L with $P_1[x(0) = x] = P_2[x(0) = x]1$. Clearly P_1, P_2 can be modified by keeping them on $\mathcal{F}_{\tau_1}$ and making $P_1|\mathcal{F}_{\tau_1}$ for $t \ge \tau_1$ to be $L^{\{\alpha(x)\}}$. Then the new measures $\hat{P}_1$ and $\hat{P}_2$ are both solutions of $L^{\{\alpha(x)\}}$. Since uniquenes holds now $\hat{P}_1 = \hat{P}_2$ and in particular $P_1 = P_2$ on $\mathcal{F}_{\tau_1}$. The conditional distributions $Q^{1,\tau,\omega}$ and $Q^{2,\tau,\omega}$ are solutions for L starting from $x(\tau_1)$ and cam ne modofied to be solutions of $L^{\{\alpha(x(\tau_1))\}}$. Follows that $Q^{1,\tau,\omega} = Q^{2,\tau,\omega}$ for almost all ω and now $P_1 = P_2$ on $\mathcal{F}_{\tau_2}$. Let us define successive stopping times $\tau_0 = 0$ and

$\tau_{j+1} = \inf\{t \geq \tau_j : |x(t) - x(\tau_j)| \geq B(x(\tau_j), \epsilon(x(\tau_j))\}$. $\tau_k = T$ for some finite k or if T is infinite $\lim_{j\to\infty} \tau_j = \infty$. Induction works and $P_1 = P_2$ on ${}_{\tau_n}$ and letting $n \to \infty$ $P_1 = P_2$.

We will prove existence and uniqueness for a class of operaors

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

Assume, $a_{i,j}$ are continuous, positive definite for each x and $b_i(x)$ are bounded and measurable. We know that for each x solution exists. It is enough to prove uniqueness assuming that $b_i(x) = 0$ and $|a_{i,j}(x) - \delta_{i,j}| < \epsilon$ for some $\epsilon > 0$. As we saw earlier it is enough to solve he equation

$$\lambda u - Lu = f$$

for sufficiently many functions. For $\lambda > 0$ what is the range of $\lambda u - Lu$ as u varies over smooth functons with compact support?

Step 1. For every p there in an $\epsilon = \epsilon(p, d)$ such that the range is dense in $L_p(R^d)$, if $|a_{i,j}(x) - \delta_{i,j}| < \epsilon(p, d)$

Step 2. If P is any solution for $L = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}$ and $|a_{i,j}(x) - \delta_{i,j}| < \epsilon(p, d)$ with $P[x(0) = x] = 1$ then

$$\Lambda(\lambda, f) = E^P[\int_0^\infty e^{-\lambda t} f(x(t))dt]$$

satifies $|\Lambda(f)| \leq C\|f\|_p$.