Explosion.

We have so far assumed that a and b are bounded. Supose they are bounded on compact sets but are allowed to grow near ∞. Then existence can be in trouble. For example if $a = 0$ and $b = 1 + x^2$ in one dimension the ODE blows up. One should expect a similar behavior when a is present. Let

$$L = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_j b_j(x)\frac{\partial}{\partial x_j}$$

If there is a smooth ffunction $u \geq 1$ such that $u(x) \to \infty$ as $|x| \to \infty$ and for some $c > 0$

$$Lu \leq cu$$

then the process is defined for all times. The issue really is does the exit time $\tau_\ell = \inf\{t : |x(t)| \geq \ell\}$ from a ball of radius ℓ, which can be infinite, satisfies

$$\lim_{\ell\to\infty} P[\tau_\ell \leq T] = 0$$

for every $0 < T < \infty$. It is assumed that the process P is locally well defined and is a solution for L unitl exit from any compact set $B_\ell = \{x : |x| \leq \ell\}$. The proof is very simple. For some $\alpha(t,\omega) \geq 0$

$$du(x(t))e^{-ct} = -cu(x(t))e^{-ct} + e^{-ct}(Lu)(x(t))dt + dM(t) = -\alpha(t)dt + dM(t)$$

making $e^{-ct}u(x(t))$ a super martingale. In partcular

$$\begin{aligned} u(x) &\geq E^{P_x}[e^{-c(\tau_\ell\wedge T)}u(x(\tau_\ell \wedge T))] \\ &\geq E^{P_x}[\mathbf{1}_{\tau_\ell<T}e^{-c(\tau_\ell\wedge T)}u(x(\tau_\ell \wedge T))] \\ &\geq [\inf_{|x|=\ell} u(x)]e^{-cT}P[\tau_\ell < T] \end{aligned}$$

Since $[\inf_{|x|=\ell} u(x)] \to \infty$ as $\ell \to \infty$, it follows that $\lim_{\ell\to\infty} P[\tau_\ell < T] = 0$ for every T.

If $\sum_i a_{i,i}(x) \leq (1 + |x|^2)$ and $\sum_j |b_j(x)| \leq (1 + |x|^2)^{\frac{1}{2}}$ then with $u(x) = (1 + |x|^2)^k$, Lu is dominated by a polynomial of degree $2k$ and so dominated by cu for some constant. In the other direction if there is a bounded function $u(x) > 0$ such that $Lu \geq cu$ for some $c > 0$, then the process explodes in a finite time with positive probability. To see this, note that now $e^{-ct}u(x(t))$ a bounded sub martingale.

$$\begin{aligned} u(x) &\leq E^{P_x}[e^{-c\tau_\ell}u(x(\tau_\ell))] \\ &\leq [\sup_x u(x)]E^{P_x}[e^{-c\tau_\ell}] \end{aligned}$$

Follows that

$$\liminf_{\ell\to\infty} E^{P_x}[e^{-c\tau_\ell}] \geq \frac{u(x)}{\sup_x u(x)} > 0$$

and contradicts

$$\lim_{\ell\to\infty} P_x[\tau_\ell \leq T] = 0.$$

For example is the probability that the process corresponding to $L = \frac{1}{2}\frac{\partial^2}{\partial x^2} + x^2\frac{\partial}{\partial x}$ blows up in finite time positive in $1d$?