Problems. Feb 25, 2020.

1. Let X be the Banach space $L_p[0,1]$ with Lebesgue measure. $1 < p < \infty$. Show that the dual X^* is $L_q[0,1]$ where $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L_q[0,1]$, $\Lambda_g(f) = \int_0^1 f(x)g(x)dx$. $\Lambda_n \in X^*$ converges weakly to Λ if $\Lambda_n(f) \to \Lambda(f)$ for all $f \in L_p[0,1]$. Show that if Λ_n tends weakly to Λ and $\|\Lambda_n\| \to \|\Lambda\|$ then $\|\Lambda - \Lambda\| \to 0$. What if $p = 1$ or ∞?

2. Show that the dual of the space of continuous functions on $[0,1]$ with $\|f\| = \sup_x |f(x)|$ is the space of signed measures μ of finite variation. $\Lambda(f) = \int_0^1 f(x)\mu(dx)$. $\|\Lambda\| = \sup_A [|\mu(A)| + |\mu(A^c)|]$ where A varies over all Borel sets.