Probability, Limit Theorems

Problem set 6. Due Oct 31, 2002

Q1. In the law of the iterated logarithm, where

$$\limsup \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{2n \log\log n}} = 1$$

$$\liminf \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{2n \log\log n}} = -1$$

prove that the set of limit points of

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{2n \log\log n}}$$

consists of the entire interval $[-1, 1]$.

Q2. Consider the binary expansion of a number $0 < x < 1$

$$x = \sum_{k=1}^{\infty} \frac{a_k}{2^k}$$

where $a_k = 0$ or 1.

a) Prove that a_k are well defined for almost all x (w.r.t Lebesgue measure).

b) Prove that almost surely

$$\lim_{n\to\infty} \frac{a_1 + a_2 \cdots + a_n}{n} = \frac{1}{2}$$

c) What can you say about

$$\limsup_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n - \frac{n}{2}}{\sqrt{2n \log\log n}}$$