Chapter 1

Fourier Series

We will consider complex valued periodic functions with period 2π. We can view them as functions defined on the circumference S of the unit circle in the complex plane or equivalently as function f defined on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$. The **Fourier Coefficients** of the function f are defined by

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \tag{1.1}$$

and formally

$$f(x) \simeq \sum a_n e^{inx} \tag{1.2}$$

If we assume that $f \in L_1[0, 2\pi]$ then clearly a_n is well defined and

$$|a_n| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| dx$$

It is not clear that the sum on right hand side of equation 1.2 converges and even if it does it is not clear that it is actually equal to the the function $f(x)$. It is relatively easy to find conditions on $f(\cdot)$ so that the sum in 1.2 is convergent. If $f(x)$ is assumed to be k times continuously differentiable on S, integrating by parts k times one gets, for $n \neq 0$,

$$|a_n| \leq \frac{1}{n^k} \sup_x |f^{\{k\}}(x)| \tag{1.3}$$

From the estimate 1.3 it is easily seen that the sum is convergent if f is twice continuosly differentiable.

Another important but elementary fact is

Theorem 1 (Riemann-Lebesgue). *For every $f \in L_1[-\pi,\pi]$,*

$$\lim_{n\to\pm\infty} a_n = 0 \tag{1.4}$$

Let us define the partial sums

$$s_N(f,x) = \sum_{|n|\le N} a_n e^{inx} \tag{1.5}$$

and the Fejer sum

$$S_N(f,x) = \frac{1}{N+1}\sum_{0\le n\le N} s_n(f,x) \tag{1.6}$$

We can calculate

$$\begin{aligned} s_n(f,x) =& \frac{1}{2\pi}\sum_{|j|\le n} e^{ijx}\int_{-\pi}^{\pi} e^{-ijy} f(y)dy \\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)[\sum_{|j|\le n} e^{ij(x-y)}]dy \\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\frac{e^{-in(x-y)}(e^{i(2n+1)(x-y)}-1)}{e^{i(x-y)}-1}dy \\ =& \int_0^{2\pi} f(y)k_n(x-y)dy \end{aligned} \tag{1.7}$$

where

$$k_n(z) = \frac{1}{2\pi}\frac{e^{-inz}(e^{i(2n+1)z}-1)}{e^{iz}-1} = \frac{1}{2\pi}\frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}} \tag{1.8}$$

A similar calculation reveals

$$S_N(f,x) = \int_{-\pi}^{\pi} f(y)K_N(x-y)dy \tag{1.9}$$

where

$$\begin{aligned} K_N(z) =& \frac{1}{2\pi}\frac{1}{(N+1)}\frac{1}{\sin\frac{z}{2}}\sum_{0\le n\le N}[e^{i(n+1)z} - e^{-inz}] \\ =& \frac{1}{2\pi}\frac{1}{(N+1)}\frac{1}{\sin\frac{z}{2}}\frac{1}{e^{iz}-1}[(e^{iz}-e^{-iNz})(e^{i(N+1)z}-1)] \\ =& \frac{1}{\pi}\frac{1}{(N+1)}\left[\frac{\sin(N+\frac{1}{2})z}{\sin\frac{z}{2}}\right]^2 \end{aligned} \tag{1.10}$$

Notice that for every N,

$$\int_{-\pi}^{\pi} k_N(z)dz = \int_{-\pi}^{\pi} K_N(z)dz = 1 \tag{1.11}$$

The following observations are now easy to make.

1. Nonnegativity.

$$K_N(z) \geq 0$$

2. For any $\delta > 0$,

$$\lim_{N\to\infty} \sup_{|z|\geq\delta} K_N(z) = 0$$

3. Therefore

$$\lim_{N\to\infty} \int_{|z|\geq\delta} K_N(z)dz = 0$$

It is now an easy exercise to prove

Theorem 2. *For any f that is bounded and continuous on S*

$$\lim_{N\to\infty} \sup_{x\in S} |S_N(f,x) - f(x)| = 0$$

Theorem 3. *For any $f \in L_p[-\pi,\pi]$*

$$\|S_N(f,\cdot)\|_p \leq \|f\|_p$$

and therefore for $1 \leq p < \infty$,

$$\lim_{N\to\infty} \|S_N(f,\cdot) - f(\cdot)\|_p = 0$$

The behavior of $s_N(f,x)$ is more complicated. However it is easy enough to see that

Theorem 4. *For $f \in C^2(S)$,*

$$\lim_{N\to\infty} \sup_{x} |s_N(f,x) - f(x)| = 0$$

The series converges and so $s_N(f,\cdot)$ has a uniform limit g. $S_N(f,\cdot)$ has the same limit, but has just been shown to converge to f. Therefore $f = g$. The following Theorem is fairly easy.

Theorem 5. *If f is a function in $C^\alpha(S)$ i.e Hölder continuous with some exponent $\alpha > 0$ then*

$$\lim_{N\to\infty} s_N(f,x) = f(x)$$

Proof. We can assume thatwith out loss of generality that $x = 0$ and let $f(0) = a$. We need to show that

$$\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}}dy = a \tag{1.12}$$

Because $\frac{f(y)-a}{\sin\frac{y}{2}}$ is integrable, 1.12 is consequence of the Riemann-Lebesgue Theorem, i.e. Theorem 1. $\square$

If f is a bounded function, then one can replace $\sin\frac{y}{2}$ by $\frac{y}{2}$ and the problem reduces to calculating

$$\lim_{\lambda\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\frac{\sin\lambda y}{y}dy$$

Let us now assume that f is a function of bounded variation on S which has left and right limits a_l and a_r at 0. By a change of variables one can reduce the above to calculating

$$\lim_{\lambda\to\infty} \frac{1}{\pi}\int_{-\lambda\pi}^{\lambda\pi} f(\frac{y}{\lambda})\frac{\sin y}{y}dy$$

If we denote by

$$G(y) = \int_y^\infty \frac{\sin x}{x}dx$$

then

$$\begin{aligned}
a_r(\lambda) &= \frac{1}{\pi}\int_0^{\lambda\pi} f(\frac{y}{\lambda})\frac{\sin y}{y}dy = -\frac{1}{\pi}\int_0^{\lambda\pi} f(\frac{y}{\lambda})dG(y) \\
&= \frac{1}{2}a_r + \frac{1}{\pi}\int_0^{\lambda\pi} G(y)df(\frac{y}{\lambda}) = \frac{1}{2}a_r + \frac{1}{\pi}\int_0^{\pi} G(\lambda y)df(y) \\
&\to \frac{1}{2}a_r
\end{aligned}$$

by the bounded convergence theorem. This establishes the following

Theorem 6. *If f is of bounded variation on S*

$$\lim_{N\to\infty} s_N(f,x) = \frac{f(x+0)+f(x-0)}{2}$$

The behavior of $s_N(f,x)$ for f in $L_p[-\pi,\pi]$ for $1 \le p < \infty$ is more complex. Let us define the linear operator

$$(T_\lambda f)(x) = \int_{-\pi}^{\pi} f(x+y)\frac{\sin\lambda y}{\sin\frac{y}{2}}dy \tag{1.13}$$

on smooth functions f. If $s_N(f,x)$ were to converge uniformly to f for every bounded continuous function it would follow by the uniform boundedness principle that

$$\sup_x |(T_\lambda f)(x)| \le C\sup_x |f(x)|$$

with a constant independent of λ, atleast for $\lambda = N+\frac{1}{2}$ for positive integers N. Let us show that this is false. The best possible bound $C = C_\lambda$ is seen to be

$$C_\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\sin\lambda y|}{|\sin\frac{y}{2}|}dy$$

and because

$$|\frac{1}{\sin\frac{y}{2}} - \frac{2}{y}|$$

is integrable, C_λ differs from

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\sin\lambda y|}{|y|}dy = \frac{1}{2\pi}\int_{-\lambda\pi}^{\lambda\pi}\frac{|\sin y|}{|y|}dy$$

by a uniformly bounded amount. The divergence of

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\sin y|}{|y|}dy$$

implies that $C_\lambda \to \infty$ as $\lambda \to \infty$. By duality theis means that $T_\lambda f$ is not uniformly bounded as an operator from $L_1[-\pi,\pi]$ into itself either. Again from uniform boudedness principle one cannot expect that $s_N(f,\cdot)$ tends to

$f(\cdot)$ in $L_1[-\pi,\pi]$ for evrery $f \in L_1[-\pi,\pi]$. However we will prove that for $1 < p < \infty$, for $f \in L_p[-\pi,\pi]$

$$\lim_{\lambda\to\infty} \|T_\lambda f - f\|_p = 0$$

By standard arguments involving the approximation of an L_p function by a continuous function it is sufficient to prove a uniform bound of the form

$$\|T_\lambda f\|_p \le C_p \|f\|_p$$

with a constant C_p depending only on p for smooth functions f and $\lambda \ge 1$ First we establtsh what is known as weak type inequality.

Theorem 7. *There is a constant C such that for all $\lambda \ge 1$ and smooth f*

$$mes\{x : |(T_\lambda f)| \ge \ell\} \le \frac{C}{\ell}\|f\|_1$$