12 Representations SO(3)

We will consider the irreducible representations of the group G of rotations in R^3. These are orthogonal transformations of determinant 1, i.e. that preserve orientation. An element $g \in G$ is represented as the matrix

$$\begin{bmatrix} t_{1,1}(g) & t_{1,2}(g) & t_{1,3}(g) \\ t_{2,1(g)} & t_{2,2}(g) & t_{2,3}(g) \\ t_{3,1(g)} & t_{3,2}(g) & t_{3,3}(g) \end{bmatrix}$$

There is the trivial representation $\pi_0(g) \equiv I$. Then there is a natural three dimensional representation where $\pi_1(g) = t(g) = \{t_{i,j}(g)\}$ and it can be viewed as a unitary representation in $\mathcal{C}^3$. This representation is irreducible and faithful, i.e. it seperates points of G.

As we saw in the general theory, the characters can be used to identify the irreducible representations. It helps to know what the conjugacy classes are. Given two orthogonal matrices g_1 and g_2, when can we find a g such that $gg_1g^{-1} = g_2$? The eigen values of g_1 are $1, e^{\pm i\theta_1}$ and therefore in order for g_1 and g_2 to be mutually conjugate we neeed $\theta_1 = \pm\theta_2$ or $cos\theta_1 = \cos\theta_2$. Conversely one can show that that if g_1 and g_2 have the same eigenvalues then they are indeed conjugate. If we use a g to align the eigenspace correponding to 1, then we need to show essentially that rotation by θ and $-\theta$ are conjugate. We can use the matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

to achieve this.

We will use the infinitesimal method to study irreducible representations. If $A = \{a_{i,j}\}$ is a real skewsymmetric matrix then $g_t = e^{tA}$ defines a one parameter curve in G, and if π is a unitary representation on a complex vector space V, then $U_t = \pi(g_t) = e^{it\sigma(A)}$ for some skew symmetric $\sigma(A)$. This way we get a map $A \to \sigma(A)$ from the space of real skewsymmetric 3×3 matrices into complex skewhermitian matrices on V.

The way to understand this map is to think of G as three dimensional manifold and the vector space of real skewsymmetric 3×3 matrices as the tangent space at e. In fact there are global vector fields acting on functions defined on G corresponding to any skew symmetric A,

$$(X_A)f(g) = \frac{d}{dt} f(ge^{tA})|_{t=0}$$

Then

$$\sigma(A) = (X_A)\pi(e)$$

and from the representation property

$$(X_A)\pi(g) = \pi(g)\sigma(A)$$

$$X_A X_B = \sigma(A)\sigma(B)$$

The Poisson bracket $[X_A, X_B] = X_A X_B - X_B X_A$ is to equal $X_{[AB-BA]}$ and we get this a way a representation σ of the "Lie Algebra" of 3×3 skewsymmetric matices in the space of skewhermitian trnasfromations on V. Moreover $\sigma([A, B]) = [\sigma(A), \sigma(B)]$. G acts irreducibly on V if and only if $\sigma(A)$ acts irreducibly. We pick a basis A_1, A_2, A_3 where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let us note that

$$[A_1, A_2] = -A_3, [A_2, A_3] = -A_1, [A_3, A_1] = -A_2$$

If we define $\sigma(A_1) = H$ and $Z_1 = \sigma(A_2) + i\sigma(A_3), Z_2 = \sigma(A_2) - i\sigma(A_3)$, we can calculate

$$\begin{aligned} [H, Z_1] &= \sigma([A_1, A_2]) + i\sigma([A_1, A_3]) = -\sigma(A_3) + i\sigma(A_2) = iZ_1 \\ [H, Z_2] &= \sigma([A_1, A_2]) - i\sigma([A_1, A_3]) = -\sigma(A_3) - i\sigma(A_2) = -iZ_2 \end{aligned}$$

H being skewhermitian on V, it has purely imaginary eigenvalues and a complete set of eigenvectors. Let $V = \oplus_\lambda V_{i\lambda}$ be the decomposition of V into eigenspaces of H. Moreover $e^{2\pi H} = \pi(e^{2\pi A_1}) = \pi(e) = I$ The values λ are therefore all integers. If $Hv = i\lambda v$, then $HZ_1 v = Z_1 H v + [H, Z_1]v = i\lambda Z_1 v + iZ_1 v = i(\lambda + 1)Z_1 v$. Therefore Z_1 maps $V_{i\lambda} \to V_{i(\lambda+1)}$ and similarly Z_2 maps $V_{i\lambda} \to V_{i(\lambda-1)}$. It is clear that if we start with some $v_0 \in V_{i\lambda}$ then $v_0, \{Z_1^k v_0 : k \geq 1\}, \{Z_2^k v_0 : k \geq 1\}$ are all mutually orthogonal. Since the space is finite dimensional, $Z_1^r v_0 = Z_2^s v_0 = 0$ for some r, s. If we take r, s to be the smallest such values, then the subspace generated by them has dimension $r+s-1$ and is invariant under H, Z_1, Z_2. Since the representation is irreducible, this must be all of V. Another piece of information is that H and $-H$ are conjugate. The set of λ's is therefore symmetric around the origin. Hence V is odd dimensional and is $\{\lambda\} = \{-k, \ldots, 0, \ldots, k\}$ for some integer $k \geq 0$. This exhausts all possible irreducible representations in the infinitesimal sense and therefore the set of irreducible representations of G cannot be larger. The character of such a representation if it exists is seen to be

$$\chi_k(g) = \hat{\chi}_k(\theta) = \sum_{j=-k}^{k} \exp[i\, j\theta]$$

where $1, e^{\pm i\theta}$ are the eigenvalues of g. We will try construct them as the natural action of G on the space of homogeneous harmonic polynomials of degree k. This dimension is calculated as $\frac{(k+1)(k+2)}{2} - \frac{k(k-1)}{2} = 2k+1$. H which is the infinitesimal rotation around x-axis is calculated as

$$H = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$

The polynomials $p_k^{\pm} = (y \pm iz)^k$ are harmonic in two and therefore three variables and $Hp_k^{\pm} = \pm ikp_k^{\pm}$. Therefore this representation has the eigenvlaues $\pm ik$ for H and cannot be decomposed totally in terms of representations of dimension $(2k-1)$ or less. On the other hand its dimension is only $(2k+1)$. This is it.

Since we know that $\chi_k(g)\chi_\ell(g)dg = \delta_{k,\ell}$ it is convenient to determine the weight $w(\theta)$ on $[0, \pi]$ such that it is the probability density of $\theta(g)$ of a random g. Then

$$\int_0^{\pi} \hat{\chi}_k(\theta)\hat{\chi}_\ell(\theta)w(\theta)d\theta = \delta_{k,\ell}$$

In particular for $k \geq 2$

$$\int_0^{\pi} [\hat{\chi}_k(\theta) - \hat{\chi}_{k-1}(\theta)]w(\theta)d\theta = \delta_{k,\ell}$$

or

$$w(\theta) = a + b\cos\theta$$

Normalization of $\int_0^{\pi} w(\theta)d\theta = 1$ gives $a = \frac{1}{\pi}$. The orthogonality relation $\int_0^{\pi} 1.(1 + 2\cos\theta)w(\theta)d\theta = 0$ provides $a + b = 0$ or

$$w(\theta) = \frac{1 - \cos\theta}{\pi}$$