4 Riesz Kernels.

A natural generalization of the Hilbert transform to higher dimension is mutiplication of the Fourier Transform by homogeneous functions of degree 0, the simplest ones being

$$\widehat{R_i f}(\xi) = \frac{\xi_i}{|\xi|}\hat{f}(\xi) \tag{4.1}$$

Since the functions $k_i(\xi) = \frac{\xi_i}{|\xi|}$ are bounded functions it is clear that R_i are bounded operators from $L_2(R^d)$ into $L_2(R^d)$. On the other hand k_i are not continuous at $\xi = 0$, and therefore the formal kernel K_i with the representation

$$R_i f(x) = \int_{R^d} K_i(x-y) f(y) dy \tag{4.2}$$

can not be in $L_1(R^d)$.

Lemma 1. *The kernels $K_i(\cdot)$ are given by*

$$K_i(x) = c_d \frac{x_i}{|x|^{d+1}} \tag{4.3}$$

where c_d is a constant depending on the dimension.

Proof. We will begin with the following calculation. For any $\epsilon > 0, d > \delta > 0$

$$\begin{aligned}\int_{R^d} e^{i<x,\xi>} \frac{1}{|x|^{d-\delta}} e^{-\epsilon|x|^2} dx &= \frac{1}{\Gamma(\frac{d-\delta}{2})} \int_{R^d}\int_0^\infty e^{i<x,\xi>} t^{\frac{d-\delta}{2}-1} e^{-(t+\epsilon)|x|^2} dx dt \\ &= \frac{c_d}{\Gamma(\frac{d-\delta}{2})} \int_0^\infty e^{-\frac{|\xi|^2}{4(t+\epsilon)}} t^{\frac{d-\delta}{2}-1} (t+\epsilon)^{-\frac{d}{2}} dt \end{aligned} \tag{4.4}$$

□

If we let $\epsilon \to 0$ in equation (**??**)

$$\lim_{\epsilon\to 0} \int_{R^d} e^{i<x,\xi>} \frac{1}{|x|^{d-\delta}} e^{-\epsilon|x|^2} dx = \frac{c_d}{\Gamma(\frac{d-\delta}{2})} \int_0^\infty e^{-\frac{|\xi|^2}{4t}} t^{-\frac{\delta}{2}-1} dt = \frac{c_d \Gamma(\frac{\delta}{2})}{\Gamma(\frac{d-\delta}{2})} |\xi|^{-\delta}$$

If we let $f_{\epsilon,\delta}(x) = \frac{(d-\delta)x_j}{|x|^{d+2-\delta}} e^{-\epsilon|x|^2}$, $\lim_{\epsilon\to 0} \widehat{f}_{\epsilon,\delta}(x) = ic_d \frac{\Gamma(\frac{\delta}{2})}{\Gamma(\frac{d-\delta}{2})} \xi_j |\xi|^{-\delta}$. Finally we let $\delta > 1 \to 1$.

It is not difficult to see that for any smooth function $f(x)$ with compact support

$$(R_j f)(x) = \int_{|y|\le \ell} \frac{y_j}{|y|^{d+1}} [f(x+y) - f(x)] dy + \int_{|y|\ge \ell} \frac{y_j}{|y|^{d+1}} f(x+y) dy$$

is independent of ℓ because $\int_S \frac{y_j}{|y|^{d+1}} dy = 0$ for any shell $S = \{\ell_1 \le |y| \le \ell_2\}$. It is a smooth function of x. For large x, the first term is 0, and the second integral can be estimated by,

$$\int_{R^d} \Big| \Big[\frac{x_j - y_j}{|y-x|^{d+1}} - \frac{x_j}{|x|^{d+1}} \Big] f(y) \Big| dy \le \frac{C}{|x|^{d+1}}$$

if we use that f has compact support and satisfies $\int_{R^d} f(y) dy = 0$. It is now easy to compute

$$\widehat{R_j f}(\xi) = \frac{1}{c_d} \frac{\xi_j}{|\xi|} \hat{f}(\xi)$$

The next step is to show that the kernels K_i satisfy condition of equation (**??**). Let us take $x, y \in R^d$ and write $y = r\omega$ where $r = |y|$ and $\omega = \frac{y}{|y|} \in S^{d-1}$.

$$\begin{aligned}
\int_{|x-y|\ge C|y|} \Big| \frac{x_i - y_i}{|x-y|^{d+1}} - \frac{x_i}{|x|^{d+1}} \Big| dx &= \int_{|x-y|\ge C|y|} \Big| \frac{\sigma(x-y)}{|x-y|^d} - \frac{\sigma(x)}{|x|^d} \Big| dx \\
&\le \int_{|x-r\omega|\ge Cr|} \Big| \frac{\sigma(x-r\omega)}{|x-r\omega|^d} - \frac{\sigma(x-r\omega)}{|x|^d} \Big| dx \\
&\quad + \int_{|x-r\omega|\ge Cr} \frac{|\sigma(x-r\omega) - \sigma(x)|}{|x|^d} dx \\
&\le C_1 \int_{|x-r\omega|\ge Cr} \Big| \frac{1}{|x-r\omega|^d} - \frac{1}{|x|^d} \Big| dx \\
&\quad + \int_{|x-r\omega|\ge Cr} \frac{|\sigma(x-r\omega) - \sigma(x)|}{|x|^d} dx
\end{aligned}$$

where $\sigma(x) = \frac{x_i}{|x|}$. If we make the substitution $x = rx'$ we get

$$\int_{|x-y|\geq C|y|} |\frac{x_i - y_i}{|x-y|^{d+1}} - \frac{x_i}{|x|^{d+1}}|dx \leq C_1 \int_{|x'-\omega|\geq C} |\frac{1}{|x'-\omega|^d} - \frac{1}{|x'|^d}|dx' + \int_{|x'-\omega|\geq C} \frac{|\sigma(x'-\omega) - \sigma(x')|}{|x'|^d} dx'$$

The estimate is clearly uniform in r. If C is large enough 0 and ω are excluded from the domain of integrartion. For large x we get an extra cancellation in both the integrals to make them converge with a bound that is uniform in ω. For the second integral we need only that σ satisfies a Hölder condition on S^{d-1}.

We have therefore proved the following theorem.

Theorem 4.1. *If the kernel $K(x)$ is given by*

$$K(x) = \frac{\sigma(\frac{x}{|x|})}{|x|^d} \tag{4.5}$$

and $\sigma(\cdot)$ satisfies a Hölder condition on S^{d-1} and has mean 0 on S^{d-1}, then convolution by K defines a bounded operator from $L_p(R^d)$ into $L_p(R^d)$ for all p in the range $1 < p < \infty$. In particular the Riesz transforms 4.1 given by 4.2 with kernels 4.3 are bounded operators in every L_p in the same range.

5 Sobolev Spaces.

In dealing with differential equations we often come across solutions that do not have the smoothness necessary to be a solution in the ordinary sense. To illustrate it by an example, suppose we want to solve the equation

$$\Delta u = \sum_i u_{x_i x_i} = f \tag{5.1}$$

on R^d. If $d = 1$ the equation reduces to $u_{xx} = f$ which is easy to solve. We need only to integrate f twice, and if f has d continuous derivatives u will have $d+2$ continuous derivatives. On R^d it is conceivable that each $u_{x_i x_i}$ may be singular, but somehow the singularities cancel miraculously to produce a much nicer f. Working formally with Fourier trnasforms

$$-|\xi|^2 \widehat{u}(\xi) = \widehat{f}(\xi)$$

and

$$\widehat{u}_{x_ix_j}(\xi) = -\frac{\xi_i\xi_j}{|\xi|^2}\widehat{f}(\xi)$$

In other words

$$u_{x_ix_j} = -R_iR_jf$$

It says that for $1 < p < \infty$, if $f \in L_p$ we can expect u to have two derivatives in L_p, but if f is bounded and continuous one should not expect u to have two continuous derivatives. In fact on $d = 2$, one can construct a counter example, i.e. a function f which is continuous such that the soulution u of Poisson's equation exhibits a singularity of the individual second derivatives at 0, that of course cancel to produce a continuous f.

The Sobolev spaces $W^p_k(R^d)$ are defined as the space of functions u on R^d such that u and all its partial derivatives $D^{n_1}_{x_1}\cdots D^{n_d}_{x_d}u$ of order $n = n_1 + \cdots + n_d$ are in L_p. We could start with C^∞ functions with compact support on R^d and complete it in the norm

$$\|u\|_{k,p} = \sum_{\substack{n_1,\cdots n_d \\ n=n_1+\cdots+n_d\le k}} \|D^{n_1}_{x_1}\cdots D^{n_d}_{x_d}u\|_p \tag{5.2}$$

If $u \in L_p$ and $D_iu = D_{x_i}u \in L_p$ u should be more regular than an L_p function.

Let us consider the operator

$$\widehat{Au}(\xi) = \frac{1}{(1+|\xi|^2)^{\frac{1}{2}}}\,\widehat{u}(\xi)$$

and consider its represenation by the kernel

$$(Au)(x) = \int_{R^d} u(x+y)a(y)dy$$

where

$$\begin{aligned} a(x) &= c_d\int_{R_d}\frac{e^{-i<x,\xi>}}{(1+|\xi|^2)^{\frac{1}{2}}}dx = \frac{c_d}{\sqrt{\pi}}\int_{R^d}\int_0^\infty e^{-i<x,\xi>}e^{-t(1+|\xi|^2)}\frac{1}{\sqrt{t}}dt \\ &= k_d\int_0^\infty \frac{e^{-t}}{t^{\frac{d+1}{2}}}e^{-\frac{|x|^2}{4t}}dt = \frac{k_d}{|y|^{d-1}}\int_0^\infty e^{-t\,|x|^2}e^{-\frac{1}{4t}}\frac{dt}{t^{\frac{d+1}{2}}} \end{aligned}$$

decays very rapidly at ∞, is smooth for $x \neq 0$ and has a singularity of $|x|^{1-d}$ near the origin for $d \geq 2$ and a logarithmic singularity at 0 when $d = 1$. In particular $a(\cdot) \in L_q$ for $q < \frac{d}{d-1}$. By Hölder's inequality, A will map L_p into L_∞ for $p > d$. If $d = p > 1$ the result is false. Let us take $d = 2$ and a nonnegative function f with compact support such that $f \in L_2$ but $\int_{R^d} \frac{f(x)}{|x|} dx = \infty$. We saw that Af has a singularity at 0. Let us consider $u = D_1(Af)$. Clearly

$$\|u\|_2^2 = \|\hat{u}\|_2^2 = \int_{R^2} \frac{\xi_1^2}{1+|\xi|^2} |\hat{f}(\xi)|^2 d\xi \leq \|\hat{f}\|_2^2 = \|f\|_2^2$$

By Young's inequality any $K \in L_q$ maps $L_p \to L_{p'}$ provided $\frac{1}{p} - \frac{1}{p'} = 1 - \frac{1}{q}$. Therefore $f \in W_{1,p}$ implies $f \in L_{p'}$ so long as $\frac{1}{p} - \frac{1}{p'} < \frac{1}{d}$. By induction $f \in W_{k,p}$ implies that $f \in W_{1,p}$ implies $f \in L_{p'}$ so long as $\frac{1}{p} - \frac{1}{p'} < \frac{k}{d}$). Therefore on R^d, $f \in W_{k,p}$ implies the continuity of f if $k > \frac{d}{p}$.

Actually one can prove a stronger result to the effect that if $\frac{1}{p} - \frac{1}{p'} = \frac{1}{d}$. then $W_{1,p} \subset L_{p'}$ as long as $1 < p' < \infty$. This requires the following theorem.

Theorem 5.1. *Let T_a be the operator of convolution by the kernel $|x|^{a-d}$ on R^d.*

$$(T_a f)(x) = \int_{R^d} |y|^{a-d} f(x+y) dy \tag{5.3}$$

Then T_a is bounded from L_p to $L_{p'}$ provided $1 < p < \frac{d}{a}$ and $\frac{1}{p'} = \frac{1}{p} - \frac{a}{d}$.

Proof. First, we note that for $a > 0$, T_a is well defined on bounded functions with compact support. We start by proving a weak type inequlity of the form

$$\mu[x : |(T_a f)(x)| \geq \ell] \leq C \frac{\|f\|_p^q}{\ell^q}$$

For any choice of $1 < p < \frac{d}{a}$ let $f \in L_p$. We can assume without loss of generality that $f \geq 0$. We write

$$\begin{aligned}(T_a f)(x) &= \int_{|y| \leq \rho} |y|^{a-d} f(x+y) dy + \int_{|y| \geq \rho} |y|^{a-d} f(x+y) dy \\ &\leq u_1 + u_2\end{aligned}$$

and estimate u_1, u_2 by

$$\|u_1\|_p \le C_1\rho^a\|f\|_p$$
$$\|u_2\|_\infty \le \left(\int_{|y|\ge\rho} |y|^{p^*(a-d)}dy\right)^{\frac{1}{p^*}} \|f\|_p = C_2\rho^{a-d+\frac{d}{p^*}}\|f\|_p$$

We can now pick $\rho = (\frac{2C_2\|f\|_p}{\ell})^{\frac{p}{d-ap}}$ and estimate

$$\begin{aligned}\sup_x u_2(x) &\le \frac{\ell}{2}\\ \mu[x : u_1(x) \ge \frac{\ell}{2}] &\le 2^p C_1^p \rho^{a\,p}\frac{\|f\|_p^p}{\ell^{\,p}}\\ &= C_3\left(\frac{\|f\|_p}{\ell}\right)^{\frac{ap^2}{d-ap}+p}\\ &= C_3\left(\frac{\|f\|_p}{\ell}\right)^{q}\end{aligned}$$

where $q = \frac{pd}{d-ap}$ or $\frac{1}{q} = \frac{1}{p} - \frac{a}{d}$. □

Now, an application of Marcinkiewicz interpolation gives boundedness from L_p to L_q in the same range and with the same relation between p and q.

We can also define the fractional derivative operarors

$$(|D|^a f)(x) = \int_{R^d} \frac{f(x+y) - f(x)}{|y|^{d+a}}dy \tag{5.4}$$

for $0 < a < 2$. A calculation shows that in terms of Foirier transforms it is multiplication by

$$\int_{R^d} \frac{e^{i<\xi\,,\,y>} - 1}{|y|^{d+a}}dy = c_{d,a}|\xi|^a$$

Therefore $|D|^a$ and T_a are essentially (upto a constant) inverses of each other. If $r > 0$ is written as $k + a$, where k is a nonnegative integer and $0 \le a < 1$, then one defines the norm corresponding to r^{th} derivative by

$$\|u\|_{r,p} = \sum_{\sum_i n_i \le k} \|D_1^{n_1}\cdots D_d^{n_d}u\|_p + \sum_{\sum_i n_i = k} \|D_1^{n_1}\cdots D_d^{n_d}u\|_{a,p} \tag{5.5}$$

This way the Sobolev spaces $W_{r,p}$ are defined for fractional derivatives as well.

Theorem 5.2. *The inclusion map is well defined and bounded from $W_{r,p}$ into $W_{s,q}$ provided $s < r$, $1 < p < q < \infty$, and $\frac{1}{q} \geq \frac{1}{p} - \frac{r-s}{d}$. The extreme value of $q = \infty$ is allowed if $\frac{1}{q} > \frac{1}{p} - \frac{r-s}{d}$.*

Proof. We can assume without loss of generality that $0 < r - s < 1$. We can go from $W_{r,p}$ to $W_{s,q}$ in a finite number of steps, with $0 < r - s < 1$ at each step. We write $\mathcal{I} = c_{d,a} T_a |D|^a$ where $a = r - s$. By definition $|D|^a$ maps $W_{r,p}$ boundedly into $W_{s,p}$. By the earlier theorem T_a maps $W_{s,p}$ boundedly into $W_{s,q}$. Although we proved it for $s = 0$, it is true for every s because T_a commutes with $|D|^a$. The cae $q = \infty$ is covered as well by this argument. ☐

6 Generalized Functions.

Let us begin with the space $W_{1,2}$. This is a Hilbert Space with the inner product

$$< f, g >_1 = \int_{R^d} [f\bar{g} + \sum_1^d f_{x_i} \bar{g}_{x_i}] dx = \int_{R^d} f\bar{h} dx$$

where $h = g - \sum_1^d g_{x_i x_i}$. Since $g \in W_{1,2}$, $g_{x_i} \in L_2$ and $g_{x_i x_i}$ is the derivative of an L_2 function. In fact since we can write $\int f g_{x_i} dx$ as $-\int f_{x_i} g dx$, Any derivative of an L_2 function can be thought of as a bounded linear functional on the space $W_{1,2}$. A simlar reasoning applies to all the spaces $W_{r,p}$. The dual space of $W_{r,p}$ is $W_{-r,q}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

For a function to be in L_p its singularities as well as decay at ∞ must be controlled. We can get rid of the condition at ∞ by demaniding that f be in $L_p(K)$ for every bounded set K or equivalently by insisting that $\phi f \in L_p$ for every C^∞ function ϕ with compact support. This definition makes sense for $W_{r,p}$ as well. We say that $f \in W_{r,p}^{loc}$ if $\phi f \in W_{r,p}$ for every C^∞ function ϕ with compact support. One needs to check that on $W_{r,p}$ mutiplication by a smooth function is a bounded linear map. One can use Leibnitz's rule if r is an integer. For $0 < r < 1$ we need the following lemma.

Lemma 2. *If $f \in W_{r,p}$ and $\phi \in C^{r'}$ with $r < r' \leq 1$ i.e. ϕ is a bounded function satisfying $|\phi(x) - \phi(y)| \leq C|x - y|^{r'}$, for all x, y, then $\phi f \in W_{r,p}$.*

Proof. We need to prove

$$g(x) = \int_{R^d} \frac{\phi(y)f(y) - \phi(x)f(x)}{|y - x|^{d+r}} dy$$

is in L_p. We can write

$$\phi(y)f(y) - \phi(x)f(x) = \phi(x)[f(y) - f(x)] + [\phi(y) - \phi(x)]f(y).$$

The contribution of first term is easy to control. To control the second term it is sufficient to show that

$$\sup_x \int_{R^d} \frac{|\phi(y) - \phi(x)|}{|y - x|^{d+r}} dy < \infty$$

which is not hard. We split the integral into two regions $|x - y| \leq 1$ and $|x - y| > 1$, use the Hölder property of ϕ to obtain an estimate on the integral over $|x - y| \leq 1$ and the boundedness of ϕ to get an estimate over $|x - y| > 1$, both of which are uniform in x. $\square$