7 Hardy Spaces.

For $0 < p < \infty$, the Hardy Space $\mathcal{H}_p$ in the unit disc D with boundary $S = \partial D$ consists of functions $u(z)$ that are analytic in the disc $\{z : |z| < 1\}$, that satisfy

$$\sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^p d\theta < \infty \tag{7.1}$$

From the Poisson representation formula, valid for $1 > r' > r \ge 0$

$$u(re^{i\theta}) = \frac{r'^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(r'e^{i(\theta-\varphi)})}{r'^2 - 2rr'\cos\varphi + r^2} d\varphi \tag{7.2}$$

we get the monotonicity of the quantity $M(r) = \int_0^{2\pi} |u(re^{i\theta})|^p d\theta$, which is obvious for $p = 1$ and requires an application of Hölder's inequality for $p > 1$. Actually $M(r)$ is monotonic in r for $p > 0$. To see this we note that $g(re^{i\theta}) = \log|u(re^{i\theta})|$ is subharmonic and therefore, using Jensen's inequality,

$$\begin{aligned} &\frac{r'^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\exp[pg(r'e^{i(\theta-\varphi)})]}{r'^2 - 2rr'\cos\varphi + r^2} d\varphi \\ &\qquad \ge \exp[p\frac{r'^2 - r^2}{2\pi} \int_0^{2\pi} \frac{g(r'e^{i(\theta-\varphi)})}{r'^2 - 2rr'\cos\varphi + r^2} d\varphi] \\ &\qquad \ge \exp[pg(re^{i\theta}] \end{aligned}$$

If $1 < p < \infty$ and $u(x, y)$ is a Harmonic function in D, from the bound (7.1), we can get a weak radial limit f (along a subsequence if necessary) of $u(r'e^{i\theta})$ as $r' \to 1$. In (7.2) we can let $r' \to 1$ keeping r and θ fixed. The Poisson kernel converges strongly in L_q to

$$\frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\varphi + r^2}$$

and we get the representation (7.2) for $u(re^{i\theta})$ (with $r'=1$) in terms of the boundary function f on S.

$$u(re^{i\theta}) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{f(e^{i(\theta-\varphi)})}{1^2 - 2r\cos\varphi + r^2}d\varphi \qquad (7.3)$$

Now it is clear that actually

$$\lim_{r\to 1} u(re^{i\theta}) = f(\theta)$$

in L_p. Since we can consider the real and imaginary parts seperately, these considerations apply to Hardy functions in $\mathcal{H}_p$ as well. The Poisson kernel is harmonic as a function of r, θ and has as its harmonic conjugate the function

$$\frac{1}{2\pi}\frac{2R\sin\theta}{1 - R\cos\theta + R^2}$$

with $R = \frac{r}{r'}$. Letting $R \to 1$, the imaginary part is see to be given by convolution of the real part by

$$\frac{1}{2\pi}\frac{2\sin\theta}{2(1-\cos\theta)} = \frac{1}{2\pi}\cot\frac{\theta}{2}$$

which tells us that the real and imaginary parts at any level $|z| = r$ are related through the Hilbert transform in θ. We need to normalize so that Im $u(0) = 0$. It is clear that any function in the Hardy Spaces is essentially determined by the boundary value of its real (or imaginary part) on S. The conjugate part is then determined through the Hilbert transform and to be in the Hardy class $\mathcal{H}_p$, both the real and imaginary parts should be in $L_p(R)$. For $p > 1$, since the Hilbert transform is bounded on L_p, this is essentially just the condition that the real part be in L_p. However, for $p \leq 1$, to be in $\mathcal{H}_p$ both the real and imaginary parts should be in L_p, which is stronger than just requiring that the real part be in L_p.

We prove a factorization theorem for functions $u(z) \in \mathcal{H}_p$ for p in the range $0 < p < \infty$.

Theorem 7.1. *Let $u(z) \in \mathcal{H}_p$ for some $p \in (0,\infty)$. Then there exists a factorization $u(z) = v(z)F(z)$ of u into two analytic functions v and F on D with the following properties. $|F(z)| \leq 1$ in D and the boundary value $F^*(e^{i\theta}) = \lim_{r\to 1} F(re^{i\theta})$ that exists in every $L_p(S)$ satisfies $|F^*| = 1$ a.e. on S. Moreover F contains all the zeros of u so that v is zero free in D.*

Proof. Suppose u has just a zero at the origin of order k and no other zeros. Then we take $F(z) = z^k$ and we are done. In any case, we can remove the zero if any at 0 and are therefore free to assume that $u(z) \neq 0$. Suppose u has a finite number of zeros, $z_1, \dots, z_n$. For each zero z_j consider $f_{z_j}(z) = \frac{z-z_j}{1-z\bar{z}_j}$. A simple calculation yields $|z-z_j| = |1-z\bar{z}_j|$ for $|z| = 1$. Therefore $|f_{z_j}(z)| = 1$ on S and $|f_{z_j}(z)| < 1$ in D. We can write $u(z) = v(z)\Pi_{i=1}^n f_{z_j}(z)$. Clearly the factorization $u = Fv$ works with $F(z) = \Pi f_{z_i}(z)$. If $u(z)$ is analytic in D, we can have a countable number of zeros accumulating near S. We want to use the fact that $u \in \mathcal{H}_p$ for some $p > 0$ to control the infinite product $\Pi_{i=1}^\infty f_{z_i}(z)$, that we may now have to deal with. Since $\log|u(z)|$ is subharmonic and we can assume that $u(0) \neq 0$

$$-\infty < c = \log|u(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta})|d\theta$$

for $r < 1$. If we take a finite number of zeros $z_1, \dots, z_k$ and factor $u(z) = F_k(z)v_k(z)$ where $F_k(z) = \Pi_1^k f_{z_i}(z)$ is continuous on $D \cup S$ and $|F_k(z)| = 1$ on S, we get

$$\begin{aligned}
\log|v_k(0)| &\leq \limsup_{r\to1} \frac{1}{2\pi}\int_0^{2\pi} \log|v_k(re^{i\theta})|d\theta \\
&= \limsup_{r\to1} \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta})|d\theta \\
&\leq \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|d\theta \\
&\leq C
\end{aligned}$$

uniformly in k. In other words

$$-\sum \log|f_{z_i}(0)| \leq -\log|u(0)| + C$$

Denoting $C - c$ by C_1,

$$\sum(1-|z_j|) \leq \sum -\log|z_j| \leq C_1$$

One sees from this that actually the infinite product $F(z) = \Pi_j f_{z_j}(z)e^{-i\,a_j}$

converges. with proper phase factors a_j. We write $-z_j = |z_j|e^{-i\,a_j}$. Then

$$\begin{aligned} 1 - f_{z_i}(z)e^{-i\,a_j} &= 1 + \frac{z - z_j}{1 - z\bar{z}_j}\frac{|z_j|}{z_j} \\ &= \frac{z_j - z|z_j|^2 + z|z_j| - z_j|z_j|}{z_j(1 - z\bar{z}_j)} \\ &= \frac{(1 - |z_j|)(z_j + z|z_j|)}{z_j(1 - z\bar{z}_j)} \end{aligned}$$

Therefore $|1 - f_{z_j}(z)e^{-i\,a_j}| \leq C(1 - |z_j|)(1 - |z|)^{-1}$ and if we redefine $F_n(z)$ by

$$F_n(z) = \Pi_{j=1}^{n} f_{z_j}(z)e^{-i\,a_j}$$

we have the convergence

$$\lim_{n\to\infty} F_n(z) = F(z) = \Pi_{j=1}^{\infty} f_{z_j}(z)e^{-i\,a_j}$$

uniformly on compact subsets of D as $n \to \infty$. It follows from $|F_n(z)| \leq 1$ on D that $|F(z)| \leq 1$ on D. The functions $v_n(z) = \frac{u(z)}{F_n(z)}$ are analytic in D (as the only zeros of F_n are zeros of u) and are seen easily to converge to the limit $v = \frac{u}{F}$ so that $u = Fv$. Moreover $F_n(z)$ are continuous near S and $|F_n(z)| \equiv 1$ on S. Therefore,

$$\begin{aligned} \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |v_n(re^{i\,\theta})|^p d\theta &= \limsup_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |v_n(re^{i\,\theta})|^p d\theta \\ &= \limsup_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \frac{|u(re^{i\,\theta})|^p}{|F_n(re^{i\,\theta})|^p} d\theta \\ &= \limsup_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\,\theta})|^p d\theta \\ &= \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\,\theta})|^p d\theta \end{aligned}$$

Since $v_n(z) \to v(z)$ uniformly on compact subsets of D, by Fatou's lemma,

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |v(re^{i\,\theta})|^p d\theta \leq \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\,\theta})|^p d\theta \qquad (7.4)$$

In other words we have succeeded in writing $u = Fv$ with $|F(z)| \leq 1$, removing all the zeros of u, but v still satsfying (7.4). In order to complete the proof of the theorem it only remains to prove that $|F(z)| = 1$ a.e. on S. From (7.4) and the relation $u = vF$, it is not hard to see that

$$\lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |v(re^{i\theta})|^p (1 - |F(re^{i\theta})|^p) d\theta = 0$$

Since $F(re^{i\theta})$ is known to have a boundary limit F^* to show that $|F^*| = 1$ a.e. all we need is to get uniform control on the Lebesgue measure of the set $\{\theta : |v(re^{i\theta})| \leq \delta\}$. It is clearly sufficient to get a bound on

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |\log |v(re^{i\theta})|| d\theta$$

Since $\log^+ v$ can be dominated by $|v|^p$ with any $p > 0$, it is enough to get a lower bound on $\frac{1}{2\pi}\int_0^{2\pi} \log |v(re^{i\theta})| d\theta$ that is uniform as $r \to 1$. Clearly

$$\frac{1}{2\pi} \int_0^{2\pi} \log |v(re^{i\theta})| d\theta \geq \log |u(0)|$$

is sufficient. □

Theorem 7.2. *Suppose $u \in \mathcal{H}_p$. Then $\lim_{r\to 1} u(re^{i\theta}) = u^*(e^{i\theta})$ exists in the following sense*

$$\lim_{r\to 1} \int_0^{2\pi} |u(re^{i,\theta}) - u^*(e^{i\theta})|^p d\theta = 0$$

Moreover, if $p \geq 1$, u has the Poisson kernel representation in terms of u^*.

Proof. If $u \in \mathcal{H}_p$, according to Theorem 7.1, we can write $u = vF$ with $v \in \mathcal{H}_p$ which is zero free and $|F| \leq 1$. Choose an integer k such that $kp > 1$. Since v is zero free $v = w^k$ for some $w \in \mathcal{H}_{kp}$. Now $w(re^{i\theta})$ has a limit w^* in $L_{kp}(S)$. Since $|F| \leq 1$ and has a radial limit F^* it is clear the u has a limit $u^* \in L_p(S)$ given by $u^* = (w^*)^k F^*$. If $0 < p \leq 1$ to show convergence in the sense claimed above, we only have to prove the uniform integrability of $|u(re^{i\theta})|^p = |w(re^{i\theta})|^{kp}$ which follows from the convergence of w in $L_{kp}(S)$. If $p \geq 1$ it is easy to obtain the Poisson representation on S by taking the limit as $r \to 1$ from the representation on $|z| = r$ which is always valid. □

We can actually prove a better version of Theorem 7.1. Let $u \in \mathcal{H}_p$ for some $p > 0$, be arbitrary but not identically zero. We can start with the inequality

$$-\infty < \log|u(r_0 e^{i\theta_0})| \leq \frac{r^2 - r_0^2}{2\pi} \int_0^{2\pi} \frac{\log|u(re^{i(\theta_0-\varphi)})|}{r^2 - 2rr_0\cos\varphi + r_0^2} d\varphi \qquad (7.5)$$

where $z_0 = r_0 e^{i\theta_0}$ is such that $r_0 = |z_0| < 1$ and $|u(z_0)| > 0$. We can use the uniform integrability of $\log^+|u(re^{i\theta})|$ as $r \to 1$, and conclude from Fatou's lemma that

$$\int_0^{2\pi} \frac{|\log|u(e^{i(\theta_0-\varphi)})||}{1 - 2r_0\cos\varphi + r_0^2} d\varphi < \infty$$

Since the Poisson kernel is bounded above as well as below (away from zero) we conclude that the boundary function $u(e^{i\theta})$ satisfies

$$\int_0^{2\pi} |\log|u(e^{i\theta})||d\theta < \infty$$

We define $f(re^{i\theta})$ by the Poisson integral

$$f(re^{i\theta}) = \frac{1 - r^2}{4\pi} \int_0^{2\pi} \frac{\log|u(e^{i(\theta-\varphi)})|}{1 - 2r\cos\varphi + r^2} d\varphi$$

to be Harmonic with boundary value $\log|u(e^{i\theta})|$. From the inequality (7.5) it follows that $f(re^{i\theta}) \geq \log|u(re^{i\theta})|$ We then take the conjugate harmonic function g so that $w(\cdot)$ given by $w(re^{i\theta}) = f(re^{i\theta}) + ig(re^{i\theta})$ is analytic. We define $v(z) = e^{w(z)}$ so that $\log|v| = f$. We can write $u = Fv$ that produces a factorization of u with a zero free v and F with $|F(z)| \leq 1$ on D. Since the boundaru values of $\log|u|$ and $\log|v|$ match on S, the boundary values of F which exist must satisfy $|F| = 1$ a.e. on S. We have therefore proved

Theorem 7.3. *Any u in $\mathcal{H}_p$, with $p > 0$, can be factored as $u = Fv$ with the following properties: $|F| \leq 1$ on D, $|F| = 1$ on S, v is zero free in D and $\log|v|$, which is harmonic in D, is given by the Poisson formula in terms of its boundary value $\log|v(e^{i\theta})| = \log|u(e^{i\theta})|$ which is in $L_1(S)$. Such a factorization is essentially unique, the only ambiguity being a mutiplicatve constant of absolute value 1.*

Remark. The improvement over Theorem 7.1 is that we have made sure that $\log|v|$ is not only Harmonic in D but actually takes on its boundary value in the sense $L_1(S)$. This provides the uniqueness that was missing before. As an example consider the Poisson kernel itself.

$$u(z) = e^{\frac{z+1}{z-1}}$$

$|u(z)| < 1$ on D, $u(re^{i\theta}) \to e^{i\cot\frac{\theta}{2}}$ as $r \to 1$. Such a factor is without zeros and would be left alone in Theorem 7.1, but removed now.

There are characterizations of the factor F that occurs in $u = vF$. Let us suppose that $u \in \mathcal{H}_2$ is not identically zero.. If we denote by $\mathcal{H}_\infty$, the space of all bounded analytic functions in D, clearly if $H \in \mathcal{H}_\infty$ and $u \in \mathcal{H}_2$, then $Hu \in \mathcal{H}_2$. We denote by $\mathcal{K}$ the closure in $\mathcal{H}_2$ of Hu as H varies over $\mathcal{H}_\infty$. It is clear that $\mathcal{K} = \mathcal{H}_2$ if and only if $\mathcal{K}$ contains any and therefore all of the units i.e. invertible elements in $\mathcal{H}_2$. In any case since $u \equiv 0$ is ruled out, let us pick $a \in D, a \neq 0$ such that $|u(a)| > 0$ and take $k_a \in K$ to be the orthogonal projection of $f_a(z) = \frac{1}{1-\bar{a}z}$ in $\mathcal{K}$. Note that by Cauchy's formula for any $v \in \mathcal{H}_2$,

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{f_a(e^{i\theta})}v(e^{i\theta})d\theta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{e^{i\theta}-a}v(e^{i\theta})de^{i\theta} = v(a) \tag{7.6}$$

Then $(f_a - k_a) \perp \mathcal{K}$. Writing the orthogonality relations in terms of the boundary values, and noting that $z^n k_a \in \mathcal{K}$ for $n \geq 0$,

$$\int_0^{2\pi} [\overline{f_a(e^{i\theta}) - k_a(e^{i\theta})}]e^{in\theta}k_a(e^{i\theta})d\theta =< f_a - k_a, z^n k_a >= 0 \tag{7.7}$$

On the other hand for $n \geq 0$, since $z^n k_a \in \mathcal{H}_2$, by (7.6)

$$\int_0^{2\pi} \overline{f_a(e^{i\theta})}e^{in\theta}k_a(e^{i\theta})d\theta = 2\pi a^n k_a(a)$$

Combining with equation (7.7) we get for $n \geq 0$,

$$\int_0^{2\pi} e^{in\theta}|k(e^{i\theta})|^2 d\theta = 2\pi k_a(a)a^n$$

But $|k|^2$ is real and therefore $k_a(a)$ must be real and

$$\int_0^{2\pi} e^{in\theta}|k(e^{i\theta})|^2 d\theta = \begin{cases} 2\pi k_a(a)a^n & \text{if } n > 0 \\ 2\pi k_a(a) & \text{if } n = 0 \\ 2\pi k_a(a)\bar{a}^n & \text{if } n < 0 \end{cases}$$

This implies that $|k_a(e^{i\theta})|^2 \equiv cP_a(e^{i\theta})$ on S where P_a is the Poisson kernel. If $c = 0$, it follows that $f_a \perp \mathcal{K}$, which in turn implies by (7.6) that

$$< f_a, u >= 2\pi u(a) = 0$$

which is not possible because of the choice of a. We claim that $\{k_a H\}$ as H varies over $\mathcal{H}_2$ is all of $\mathcal{K}$. If not, let $v \in \mathcal{K}$ be such that $v \perp k_a H$ for all $H \in \mathcal{H}_2$. We have then, for $n \geq 0$, taking $H = z^n$,

$$\int_0^{2\pi} \overline{k_a(e^{i\theta})} e^{-in\theta} v(e^{i\theta}) d\theta =< v, k_a z^n >= 0$$

For $n = -m < 0$, $z^m v \in \mathcal{K}$ and

$$\int_0^{2\pi} \overline{k_a(e^{i\theta})} e^{-in\theta} v(e^{i\theta}) d\theta =< z^m v, k_a >=< z^m v, f_a >= 2\pi a^m v(a)$$

Now Fourier inversion gives

$$\begin{aligned} \overline{k_a(e^{i\theta})} v(e^{i\theta}) &= v(a) \sum_{m=1}^{\infty} a^m e^{-im\theta} = v(a) \frac{ae^{-i\theta}}{1 - ae^{-i\theta}} \\ &= c_1(a) \frac{1}{e^{i\theta} - a} = c_2(a) P_a(e^{i\theta})(e^{-i\theta} - \bar{a}) \end{aligned}$$

Multiplying by k_a and remembering that $|k_a|^2 = cP_a$, we obtain $(k_a v)(e^{i\theta}) = c_3(a)(e^{-i\theta} - \bar{a})$ This leads to

$$v(e^{i\theta}) = \frac{k_a(e^{i\theta})}{e^{-i\theta} - \bar{a}} = \frac{k_a(e^{i\theta}) e^{i\theta}}{1 - \bar{a} e^{i\theta}}$$

Therefore $v = k_a H$ with $H(z) = \frac{z}{1 - \bar{a} z} \in \mathcal{H}_2$ contradicting $v \perp H k_a$ for all $H \in \mathcal{H}_2$ and forcing v to be 0. We are nowready to prove the following theorem.

Theorem 7.4. *Let $u \in \mathcal{H}_2$ be arbitrary and nontrivial. Then 1 belongs to the span of $\{z^n u : n \geq 0\}$ if and only if*

$$\log |u(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |u(e^{i\theta})| \, d\theta \tag{7.8}$$

Proof. Let $\|p_n(z)u(z) - 1\|_{\mathcal{H}_2} \to 0$ for some polynomials $p_n(\cdot)$. Then

$$\log|p_n(0)| \le \frac{1}{2\pi}\int_0^{2\pi} \log|p_n(e^{i\,\theta})|\,d\theta$$

Since $\log|p_n(e^{i\,\theta})u(e^{i\,\theta})| \to 0$ as $n \to \infty$ in measure on S and $\log^+|p_n(e^{i\,\theta})u(e^{i\,\theta})|$ is uniformly integrable,

$$\limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \log|p_n(e^{i\,\theta})u(e^{i\,\theta})|\,d\theta \le 0$$

This implies

$$\log|u(0)| \ge \frac{1}{2\pi}\int_0^{2\pi} \log|u(e^{i\,\theta})|\,d\theta$$

The reverse inequality is always valid and we are done with one half. As for the converse, If the span of $\{z^n u : n \ge 0$ is $\mathcal{K} \subset \mathcal{H}_2$ is a proper subspace, there is k such that $u = kv$ for some $v \in \mathcal{H}_2$ with $|k|^2(e^{i\,\theta}) = cP_a(e^{i\,\theta})$, the Poisson kernel for some $a \in D$. For the Poisson kernel it is easy to verify that

$$\log|P_a(0)| < \frac{1}{2\pi}\int_0^{2\pi} \log|P_a(e^{i\,\theta})|\,d\theta$$

for any $a \in D$. Therefore we cannot have (7.8) satisfied. □

Suppose $f(e^{i\theta}) \ge 0$ is a weight that is in $L_1(S)$. We consider the Hilbert Space $H = L_2(S, f)$ of functions u that are square integrable with respect to the weight f, i.e. g such that $\int_0^{2\pi} |g(e^{i\theta})|^2 f(e^{i\theta})d\theta < \infty$. The trigonometric functions $\{e^{in\theta} : -\infty < n < \infty\}$ are still a basis for H, though they may no longer orthogonal. We define $H_k = \text{span}\{e^{i\,n\,\theta} : n \ge k\}$. It is clear the $H_k \supset H_{k+1}$ and mutiplication by $e^{\pm i\theta}$ is a unitary map $U^{\pm 1}$ of H onto itself that sends H_k onto $H_{k\pm 1}$. We are interested in calculating the orthogonal projection $e_0(e^{i\,\theta})$ of 1 into H_1 along with the residual error $\|e_1(e^{i\,\theta}) - 1\|_2^2$. There are two possibilities. Either $1 \in H_1$ in which case $H_0 = H_1$ and hence $H_k = H$ for all k, or H_0 is spanned by H_1 and a unit vector $u_0 \in H_0$ that is orthogonal to H_1. If we define $u_k = U^k u_0$, then $H = \oplus_{j=-\infty}^{\infty} u_j \oplus H_\infty$ where $H_\infty = \cap_k H_k$. In a nice situation we expect that $H_\infty = \{0\}$. However if $1 \in H_1$ as we saw $H_\infty = H$. If $f(e^{i\,\theta}) \equiv c$ then of course $u_k = e^{i\,k\,\theta}$.

Theorem 7.5. *Let us suppose that*

$$\int_0^{2\pi} \log f(e^{i\theta})d\theta > -\infty \tag{7.9}$$

Then $H_\infty = \{0\}$ *and the residual error is given by*

$$\|e_0(e^{i\theta}) - e^{i\theta}\|_2^2 = 2\pi \exp[\frac{1}{2\pi}\int_0^{2\pi} \log f(e^{i\theta})d\theta] > 0 \tag{7.10}$$

Proof. We will split the proof into several steps.

Step 1. We write $f(e^{i\theta}) = |ue^{i\theta})|^2$, where u is the boundary value of a function $u(re^{i\theta}))$ in $\mathcal{H}_2$. Note that, if this were possible. according to Theorem 7.1 one can assume with out loss of generality that $u(0) \neq 0$ and for $0 < r < 1$

$$-\infty < \log|u(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta})|d\theta$$

We can let $r \to 1$, use the domination of $\log^+|u|$ by $|u|$ and Fatou's lemma on $\log^-|u|$. We get

$$-\infty < \log|u(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta})|d\theta = \frac{1}{4\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta$$

We see that the condition (7.9) is necessary for the representation that we seek. We begin with the function $\frac{1}{2}\log f \in L_1(S)$ and construct $u(re^{i\theta})$ given by the Poisson formula

$$F(re^{i\theta}) = \frac{1-r^2}{4\pi}\int_0^{2\pi} \frac{\log f(e^{i(\theta-\varphi)})}{1-2r\cos\varphi + r^2}d\varphi$$

to be Harmonic with boundary value $\frac{1}{2}\log f$. We then take the conjugate harmonic function G so that $w(\cdot)$ given by $w(re^{i\theta}) = F(re^{i\theta}) + iG(re^{i\theta})$ is analytic. We define $u(z) = e^{w(z)}$.

$$\begin{aligned}\int_0^{2\pi} |u(re^{i\theta})|^2 d\theta &= \int_0^{2\pi} \exp[2F(re^{i\theta})]d\theta \\ &\leq \int_0^{2\pi} \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{f(e^{i(\theta-\varphi)})}{1-2r\cos\varphi + r^2}d\varphi d\theta \\ &= \int_0^{2\pi} f(e^{i\theta})d\theta\end{aligned}$$

Therefore $u \in \mathcal{H}_2$ and $\lim_{r\to 1} u(re^{i\theta}) = u(e^{i\theta})$ exists in $L_2(S)$. Clearly

$$|u(e^{i\theta})| = \exp[\lim_{r\to 1} F(re^{i\theta})] = \sqrt{f(e^{i\theta})}$$

and $f = |u|^2$ on S. It is easily seen that $u(z) = \sum_{n\geq 0} a_n z^n$ with

$$\sum_{n\geq 0} |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta$$

Step 2. Our representation has the additional property that $u(z)$ is zero free in D and satifies (7.8). Suppose $h(re^{i\theta})$ is any function in $\mathcal{H}_2$ with boundary value $h(e^{i\theta})$ with $|h| = \sqrt{f}$ that also satsifies

$$\log|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\log f d\theta$$

then

$$\log|h(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|d\theta$$

By Fatou's lemma applied to $\log^-|h|$ as $r \to 1$ we get

$$\limsup_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\log f(e^{i\theta})d\theta$$

Therefore equality holds in Fatou's lemma implying the uniform integrabilty as well as the convergence in $L_1(S)$ of $\log|h(re^{i\theta})|$ to $\frac{1}{2}\log f(e^{i\theta})$ as $r \to 1$. In particular for $0 < r < 1$,

$$\log|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|d\theta$$

and hence h is zero free in D. Consequently, for $0 \leq r < r' < 1$

$$\log|h(re^{i\theta})| = \frac{r'^2 - r^2}{2\pi}\int_0^{2\pi} \frac{\log|h(re^{i(\theta-\varphi)})|}{r'^2 - 2r'r\cos d\varphi + r^2}d\varphi$$

We can let $r' \to 1$ use the convergence of $\log|h(re^{i\theta})|$ to $\frac{1}{2}\log f$ in $L_1(S)$ to conclude

$$\log|h(re^{i\theta})| = \frac{1-r^2}{4\pi}\int_0^{2\pi} \frac{\log f(\theta-\varphi)}{1 - 2r\cos d\varphi + r^2}d\varphi$$

Therfore the representation of $f(e^{i\theta}) = |u(e^{i\theta})|^2$, with $u(e^{i\theta})$ the boundary value of $u \in \mathcal{H}_2$ that satisfies condition (7.8) is unique to within a multiplicative constant of absolute value 1. The significance of making the choice of u so that the condition (7.8) is valid, is that we can conclude that $\{z^j u(z)\}$ spans all of $\mathcal{H}_2$. $\square$

Step 3. Consider the mapping from $L_2(S, d\theta)$ into $L_2(S, f)$ that sends $g(e^{i\theta}) \to \frac{g(e^{i\theta})}{u(e^{i\theta})}$. Since the integrability of $\log f$ implies that f and therefore u is almost surely nonzero on S, this map is a unitary isomorphism. Whereas any u with $|u|^2 = f$ would be enough, our u has a special property. It is the boundary value of a function $u(re^{i\theta}) \in \mathcal{H}_2$, that satisfies (7.8). Consider $g(e^{i\theta}) = a_0$. In the isomorphism it goes over to $\frac{a_0}{u}$. Its inner product with $e^{ik\theta}$ with $k \geq 1$ is given by

$$\int_0^{2\pi} [\frac{a_0}{u(e^{i\theta})}] e^{-ik\theta} u(e^{i\theta}) \bar{u}(e^{i\theta}) d\theta = a_0 \int_0^{2\pi} \bar{u}(e^{i\theta}) e^{-ik\theta} d\theta = 0$$

This shows that $\frac{a_0}{u} \perp H_1$. We claim that the decomposition $1 = \frac{a_0}{u} + \frac{u - a_0}{u}$ is the decomposition of 1 into its components in $(H_0 \cap H_1^\perp) \oplus H_1$. The residual error is given by $2\pi |a_0|^2$ which is equal to $2\pi \exp[2u(0)]$ and agrees with (7.10). We now establish our claim to complete the proof. We need to check that $\frac{a_0}{u} \in H_0$ and $\frac{u-a_0}{u} \in H_1$. In our isomorphism $1 \to \frac{1}{u}$ and for $n \geq 0$, $z^n u \to e^{in\theta}$. We know that the span of $\{z^n u(z) : n \geq 0\}$ in $\mathcal{H}_2$ is all of $\mathcal{H}_2$. Therefore $\frac{1}{u} \in H_0$. To complete the proof of our claim we need to verify that $u - a_0$ is in the span of $\{z^n u : n \geq 1\}$. This is easy because $u - a_0 = zv(z)$ for some $v \in \mathcal{H}_2$. Finally the same agument shows that the norm of the projection of 1 onto H_k equals $2\pi \sum_{i=k}^\infty |a_i|^2$ which tends to 0 as $k \to \infty$. This proves $1 \perp H_\infty$. In fact since $U^{\pm n} H_\infty = H_\infty$, it follows that $e^{in\theta} \perp H_\infty$ for every n. Therefore $H_\infty = \{0\}$.

Consider the problem of approximating a function $f_0 \in L_2(\mu)$ by linear cominations $\sum_{j \leq -1} a_j f_j$. We assume a stationarity in the form $\rho_n = \int f_j f_{n+j} d\mu$ which is independent of j. Of course $\rho_n = \rho_{-n}$ is a positive definite function and by Bochner's theorem $\rho_n = \int_0^{2\pi} e^{in\theta} dF(\theta)$ for some nonnegative measure F on S. The object to be minimized is $\int |f_0 - \sum_{j \leq -1} a_j f_j|^2 d\mu$ over all possible $a_{-1}, \ldots, a_{-k}, \ldots$. By Bochner's theorem this is equal to

$$\inf_{\{a_j : j \leq -1\}} \int_0^{2\pi} |1 - \sum_{j \leq -1} a_j e^{ij\theta}|^2 dF(\theta)$$

If $dF(\theta) = f(\theta)d\theta$, by the reality of ρ_n, $f(\theta)$ is symmetric, and we can replace $j \le -1$ by $j \ge 1$. Then this is exactly the problem we considered. The minimum is equal to $2\pi \exp[\frac{1}{2\pi}\int_0^{2\pi} \log f(\theta)d\theta]$ and we know how to find the minimizer.

Suppose now $f(t) \ge 0$ is a weight on R with $\int_{-\infty}^{\infty} f(t)dt < \infty$. Let H_a be the span of $\{e^{ibt} : b \le a\}$ in $L_2[R, f]$. For $a < 0$, what is the projection $e_a(\cdot)$ of 1 in H_a and what is the value of $\int_{-\infty}^{\infty} |1 - e_a(t)|^2 f(t)dt$?

The natural condition on f is $\int_{-\infty}^{\infty} \frac{\log f(t)}{1+t^2}dt > -\infty$. Then the Poisson integral

$$F(x, y) = \frac{y}{2\pi}\int_{-\infty}^{\infty} \frac{f(t)}{y^2 + (x-t)^2}dt$$

defines a harmonic function F in the upper half plane $C^+ = \{(x, y) : y > 0\}$ with boundary value $\frac{1}{2}\log f$ on $R = \{(x, y) : y = 0\}$ and F can be the real part of an analytic function $w = F + iG$ on C^+. The function $u = e^w$ defines an analytic function on C^+ with boundary value u^* and $f(t) = |u^*|^2$. Moreover u^* is the Fourier transform of v in $L_2(R)$ that is supported on $(-\infty, 0)$. One can again set up an isomorphism between $L_2[R, 1]$ and $L_2[R, f]$ by sending $g \to \frac{\widehat{g}}{u^*}$ ($\widehat{g}$ is the Fourier transform of g). This maps $v \to 1$ and $v(\cdot - a) \to e^{iat}$. The projecton is seen to be the image of $v\mathbf{1}_{(-\infty,a)}(\cdot)$ with the error being $\int_a^0 |v|^2(t)dt$.

Example: Suppose $f(t) = \frac{1}{1+t^2}$. The factorization $f = |u^*|^2$ is produced by $u^*(t) = (i + t)^{-1}$. This produces $v(t) = e^t\mathbf{1}_{(-\infty,0)}(t)$. The projection is the Fourier transform of $v_a(t) = e^t\mathbf{1}_{(-\infty,a)}(t)$ which is seen to be $e^a e^{iat}$.