Linear Algebra, Fall 04.

Answers to the final.

Q1. Let

$$\begin{aligned}\Delta(x) = det\begin{pmatrix} x+1 & 1 & 0 \\ x & x & x-1 \\ x+1 & 1 & x-1 \end{pmatrix} \\ &= (x+1)[x(x-1)-(x-1)] - [x(x-1)-(x-1)(x+1)] \\ &= (x+1)(x-1)^2 - [x^2 - x - x^2 + 1] \\ &= (x+1)(x-1)^2 + (x-1) \\ &= x^2(x-1)\end{aligned}$$

$\Delta(x) = 0$ if and only if $x = 0$ or 1. Therefore If $x \neq 0, 1$ the rank is 3. If $x = 0, 1$ the rank is at most 2. Set $x = 0$. Since

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

has a 2×2 minor that is nonzero the rank is 2. Similarly if $x = 1$,

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

has rank 2 as well.

Q2. A and B are both nilpotent. B is in the canonical Jordan form. Rank of A is 1. So it has a 2 dimensional null space. A^2 must therefore have 3 dimensional null space or $A^2 = 0$. Same is true for B. So the dimensions of null spaces of powers of A and B go the same way $2, 3$. Hence they have the same Jordan form.

Q3. Diagonalize A. Let $\{d_i\}$ be the eigenvalues.

$$OAO^{-1} = D = diag\{d_1, \cdots, d_n\}$$

Cleraly

$$O^{-1}\sqrt{D}O$$

where $\sqrt{D}$ is $diag\{\sqrt{d}_1, \cdots, \sqrt{d}_n\}$ will do it. For uniqueness note that if we diagonalize B, then A is diagonalized as well with eigenvalues given by squares of the eigenvalues of B. Since B is positive semidefinite the eigen values of B have to be the positive square roots of the eigenvalues of A with the same eigenspaces.

Q4. Cleraly

$$\langle A(x+ky), (x+ky)\rangle = \langle Ax, x\rangle + 2k\langle Ax, y\rangle + k^2\langle Ay, y\rangle \geq 0$$

for all k. This yields,

$$\langle Ax, y\rangle^2 \leq \langle Ax, x\rangle\langle Ay, y\rangle$$

If $\langle Ax, x\rangle = 0$, then so is $\langle Ax, y\rangle$ for all y from the above. This means $Ax = 0$.

Q5. If A is indefinite, we can assume that it has at least one positive and one negative eigen value. Take all other components to be 0. Then

$$\langle Ax, x\rangle = \lambda_1 x_1^2 - \lambda_2 x_2^2$$

$$Ax = (\lambda_1 x_1, -\lambda_2 x_2, 0, \cdots, 0)$$

Need $\lambda_1 x_1^2 = \lambda_2 x_2^2$ and $x_1^2 + x_2^2 = 1$. Solve for x_1 and x_2.