2. First order Linear Partial Differential equations.

Suppose we have an ODE in R of the form

$$\dot{x}(t) = \frac{dx(t)}{dt} = b(x(t)) \; ; x(0) = x \tag{1}$$

Then the solution $x(t)$ can be thought of as function $F(t,x)$ of t and x that provides the value $x(t)$ of the solution as a function of t and x. Is F a smooth function of t and x? It is clear that is differentiable in t and

$$\frac{dF(t,x)}{dt} = b(F(t,x)) \tag{2}$$

How about differentiability in x? If we differentiate formally and call $G = F_x$

$$\frac{dG}{dt} = b'(F(t,x))G. \tag{3}$$

Since $F(0,x) = x$, we have $G(0,x) = 1$ or

$$G(t,x) = \exp\Big[\int_0^t b'(F(s,x))ds\Big] \tag{4}$$

How do you actually prove that F_x exists and is given by the formula (4) above ? One possibility is to do the Picard iteration

$$F_n(t,x) = x + \int_0^t b(F_{n-1}(s,x))ds \tag{5}$$

and we know that $F_n \to F$. if we can show that $\frac{\partial F_n}{\partial x} \to G$ we would be done. Denoting $\frac{\partial F_n}{\partial x}$ by G_n we can differentiate the iteration formula (5) with respect to x and get

$$G_n(t,x) = 1 + \int_0^t b'(F_{n-1}(s,x))G_{n-1}(s,x)ds \tag{6}$$

We can view (5) and (6) as just the iterartion scheme for (2) and (3). Therefore $G_n \to G$.

Suppose $u(t,x)$ is a smooth function of t and x and we consider $v(t) = u(t, F(t,x))$

$$v'(t) = u_t(t, F(t,x)) + u_x F_t(t,x) = u_t(t, F(t,x)) + u_x b(F(t,x))$$

In particular if $u_t(t,x) + b(x)u_x(t,x) \equiv 0$, then $v'(t) \equiv 0$. In other words any solution of the first order partial differential equation

$$u_t + b(x)u_x = 0 \tag{7}$$

must be constant on the "characteristics" i.e. curves $(t, x(t))$ that satisfy $\dot{x}(t) = b(x(t))$. Conversely any function that is constant along characteristics must satisfy the equation (7).

If we have a solution (7) and we know the values of $u(t,x)$ at some $t = T$ as a function $g(x)$, then we can determine the value of $u(s,x)$ for any $s < T$ by solving the ODE (2). At time s start from the point x and the solution of the ODE will end up at time T at the point $F(T-s,x)$. Clearly $u(s,x) = g(F(T-s,x))$. Actually in the case of first order linear equations we can just as easily solve the ODE backwards in time. In fact this just changes b to $-b$. Therefore the solutions of (7) are determined if we know the value of u at any one time as a function of x.

All of this makes sense if $x \in R^d$. Then $b(x) : R^d \to R^d$ and the equation (7) takes the form

$$u_t + < b(x), \nabla u >= 0.$$

Examples:

1. If we wish to solve $u_t + u_x = 0$, the solution clearly is any function of the form $u(t,x) = v(x-t)$ and if we know $u(T,x) = v(x-T) = g(x)$ then $v(x) = g(x+T)$ and $u(t,x) = g(x+T-t)$.

2*. Solve $u_t + (\cosh x)^{-1} u_x = 0$; $u(0,x) = \sinh x$ for $t < 0$ and $t > 0$.

It is interesting to consider a more general form of the equation

$$< b(x), \nabla u >= 0 \tag{8}$$

in R^d and look for a solution $u(x) : G \to R$ where $G \subset R^d$ and some boundary conditions are specified on $B \subset \partial G.$, i.e $u(x) = g(x)$ for $x \in B$. To handle this one considers the ODE

$$\dot{x}(t) = b(x(t)) \; ; x(0) = x \tag{9}$$

in R^d. Clearly any solution u of (8) will be constant on the characteristics given by (9). If every charcteristic meets B exactly once before exiting from G and the characteristic from x meets B at $\hat{x}$, then clearly $u(x) = g(\hat{x})$ is the unique solution. There is trouble when some characteristics do not hit B, or they hit B in both directions. The first trouble leads to uniquness difficulties and the second to problems in existence. There is also the problem of what is to be done if a characteristic touches B tangentially and comes back inside G. Not a very clean exit!. In the earlier version with a special time coordinate $(x_0)$the equation takes the form

$$u_0 + < b(x), \nabla u >= 0$$

$\frac{dx_0}{dt} = 1$ or $x_0(t) = x_0(0) + t$ and if the boundary is of the form $x_0 = c$ it is hit exactly once by every characteristic.

Although the ODE defining the characteristics and the first order PDE are two sides of the same problem they are dual in some sense. Existence for either one implies uniqueness for the other. This is easy to see. Because if $x(t)$ is any charcteristic from x (assume for example that we are in the situation where b is continuous and we can prove existence without uniqueness for the ODE) that exits at $\hat{x}$ and u is any solution, then $u(x) = g(\hat{x})$. If u exists for enough g's then $\hat{x}$ is unique and if $\hat{x}$ exists then $u(x)$ is determined.

It is not hard to construct trivial examples of nonuniqueness. Suppose we want to solve in some domain G the equation (8). Suppose $b \equiv 0$, then any u satisfies the equation. The characteristic are all constants that go nowhere. The equation reads $0 = 0$ and any u is a solution. Higly nonunique. One can construct a better example. Let us try to solve

$$u_t + x^2 u_x = 0$$

for $t < 0$ with $u(0, x) = 0$. If we start the trajectory at some $t < 0$ it may blow up before time 0. Solving $\dot{x} = x^2$ yields $x(s) = (c - s)^{-1}$. $x = x(t)$ yields $c = t + x^{-1}$ and the trajectory $x(s) = \frac{x}{1+x(t-s)}$. Blows up when $x > 0$ and $s = t + \frac{1}{x}$. If $t + \frac{1}{x} < 0$ or $1 + tx < 0$. it is now possible to construct a nonzero solution u. $u(t, x) = 0$ if $1 + tx \geq 0$. Otherwise we take

$$u(t, x) = f\left(\frac{x}{1 + xt}\right)$$

if $(1 + xt) < 0$. If we take f to be a nice smooth compactly supported function on $[-2, -1]$ we have an example of a nontrivial u.

There are equations that are slight modifications that can be traeted as well. For instance consider for $t < T$ and $x \in R^d$,

$$u_t + < b(x), \nabla u > + c(x)u + d(x) = 0 \; ; u(T, x) = g(x) \tag{10}$$

We use a trick. Let us add two new independent variables y and z that are one dimensional so that we now have a problem in R^{d+2}. We look for a function $U(t, x, y, z)$ satisfying

$$U_t + < b(x), \nabla_x U > + c(x)U_y + d(x)\, e^y\, U_z = 0 \; ; U(T, x, y, z) = g(x)e^y + z \tag{11}$$

If u satisfies (10) then $U(t, x, y, z) = u(t, x)e^y + z$ satisfies (11). The converse is true as well. If U solves (11) so does $U(t, x, y.z + a) - a$ for every a. But the boundary values are the same. By uniqueness it follows that $U(t, x, y, z + a) = U(t, x, y, z) + a$. Therefore $U(t, x, y, z) = z + V(t, x, y)$. The function V will satisfy the equation

$$V_t + < b(x), \nabla_x V > + c(x)V_y + d(x)e^y = 0 \; ; V(T, x, y) = g(x)e^y$$

If we let $W(t, x, y) = e^{-b}V(t, x, y + b)$, then W satisfies the same equation as V and therefore $W = V$ or $V(t, x, y) = u(t, x)e^y$ for some u and then u will satisfy (10). So let us solve (11). We need to solve the ODE

$$\dot{x}(t) = b(x(t)) \; ; \; \dot{y}(t) = c(x(t)) \; ; \; \dot{z}(t) = d(x(t))e^{y(t))}$$

If we first solve for $x(\cdot)$, then

$$y(t) = y(s) + \int_s^t c(x(s))ds$$

and

$$z(t) = z(s) + \int_s^t d(x(\sigma))e^{y(\sigma)}d\sigma$$

We can write

$$\begin{aligned} U(t,x,y,z) &= g(x(T))e^{y(T)} + z(T) \\ &= g(x(T))e^{y+\int_t^T c(x(s))ds} + z + \int_t^T d(x(s))e^{y+\int_s^T c(x(\sigma))d\sigma}ds \end{aligned}$$

Therefore

$$u(t,x) = V(t,x,0,0) = g(x(T))e^{\int_t^T c(x(s))ds} + \int_t^T d(x(s))e^{\int_s^T c(x(\sigma))d\sigma}ds$$

is the solution of (10).

Let us look at the simplest equation $u_t + au_x = 0$ for some constant a with the boundary condition $u(T,x) = f(x)$. Then $u(t,x) = f(x+a(T-t))$ is the solution. We might attempt to solve it on a grid of points $\{jh, kh)\}$ with a small h and j and k running over integers. For simplicity let us take $T = 0$. Then our equation can perhaps be approximated by

$$u((jh,kh) - u((j-1)h,kh) + a[u(jh,(k+1)h) - u(jh,kh)] = 0$$

In particular

$$u((j-1)h,kh) = (1-a)u(jh,kh) + au(jh,(k+1)h)$$

which allows us to evaluate u on $t = (k-1)h$ knowing its values on $t = kh$. We start with $k = 0$ and work backward in time steps of h. After roughly $\frac{t}{h}$ steps we should get roughly $u(t,\cdot)$ if h is small enough. Do we?

If $0 \le a \le 1$, there is no problem. It is easy to show that

$$u(-nh,0) = \sum_{r=0}^{n} \binom{n}{r}(1-a)^{n-r}a^r f(rh) \to f(at)$$

(Law of Large numbers for the Binomial!)

On the other hand if $a = 2$ it is a mess. For example the term with $r = n$ is $(-1)^n 2^n f(nh)$ which is a huge term. The answer, even if it is correct, comes about by a delicate cancellation of lots of very big terms. Very unstable both mathematically if f is not smooth and computationally if we have to round off. Our discretization is **stable** if $0 \leq a \leq 1$ and perhaps not so stable otherwise. In all of these cases $u(jh, x)$ is a weighted average of $(u(j+1)h, \cdot)$. In the stable case the weights are nonnegative. If the weights are both positive and negative then the sum of the absoulte values of the weights will exceed one and iteration may increase it geometrically. We can buy ourselves some breathing space by not insisting that the lattice spacing be equal in t and x. Our grid could be $(jh, k\delta)$. The relative sizes of δ and h to be chosen with some care.

Example:

3*. For solving the equation $u_t + b(x)u_x = 0$ starting from $T = 0$ with a value $u(T, x) = f(x)$ we can construct a difference scheme of the form

$$\frac{1}{h}[u((jh, k\delta) - u((j-1)h, k\delta)] + b(x)\frac{1}{\delta}[u(jh, (k+1)\delta) - u(jh, k\delta)] = 0.$$

When is this stable? For a given b how will you choose h and δ so that the approximation is stable?