Chapter 1

Brownian Motion

1.1 Stochastic Process

A stochastic process can be thought of in one of many equivalent ways. We can begin with an underlying probability space (Ω, Σ, P) and a real valued stochastic process can be defined as a collection of random variables $\{x(t, \omega)\}$ indexed by the parameter set $\mathbf{T}$. This means that for each $t \in \mathbf{T}$, $x(t, \omega)$ is a measurable map of $(\Omega, \Sigma) \to (\mathbf{R}, \mathcal{B}_0)$ where $(\mathbf{R}, \mathcal{B}_0)$ is the real line with the usual Borel σ-field. The parameter set often represents time and could be either the integers representing discrete time or could be $[0, T]$, $[0, \infty)$ or $(-\infty, \infty)$ if we are studying processes in continuous time. For each fixed ω we can view $x(t, \omega)$ as a map of $\mathbf{T} \to \mathbf{R}$ and we would then get a *random function* of $t \in \mathbf{T}$. If we denote by $\mathbf{X}$ the space of functions on $\mathbf{T}$, then a stochastic process becomes a measurable map from a probability space into $\mathbf{X}$. There is a natural σ-field $\mathcal{B}$ on $\mathbf{X}$ and measurability is to be understood in terms of this σ-field. This natural σ-field, called the Kolmogorov σ-field, is defined as the smallest σ-field such that the projections $\{\pi_t(f) = f(t)\,;\, t \in \mathbf{T}\}$ mapping $\mathbf{X} \to \mathbf{R}$ are measurable. The point of this definition is that a random function $x(\cdot, \omega) : \Omega \to \mathbf{X}$ is measurable if and only if the random variables $x(t, \omega) : \Omega \to \mathbf{R}$ are measurable for each $t \in \mathbf{T}$.

The mapping $x(\cdot, \cdot)$ induces a measure on $(\mathbf{X}, \mathcal{B})$ by the usual definition

$$Q(A) = P\big[\omega : x(\cdot, \omega) \in A\big] \tag{1.1}$$

for $A \in \mathcal{B}$. Since the underlying probability model (Ω, Σ, P) is irrelevant, it can be replaced by the *canonical* model $(\mathbf{X}, \mathcal{B}, Q)$ with the special choice of $x(t, f) = \pi_t(f) = f(t)$. A stochastic process then can then be defined simply as a probability measure Q on $(\mathbf{X}, \mathcal{B})$.

Another point of view is that the only relevant objects are the joint distributions of $\{x(t_1, \omega), x(t_2, \omega), \cdots, x(t_k, \omega)\}$ for every k and every finite subset $F = (t_1, t_2, \cdots, t_k)$ of $\mathbf{T}$. These can be specified as probability measures μ_F on $\mathbf{R}^k$. These $\{\mu_F\}$ cannot be totally arbitrary. If we allow different permutations of the same set, so that F and F' are permutations of each other then μ_F and $\mu_{F'}$ should be related by the same permutation. If $F \subset F'$, then we can obtain the joint distribution of $\{x(t\,,\omega)\,;\, t \in F\}$ by projecting the joint distribution of $\{x(t\,,\omega)\,;\, t \in F'\}$ from $\mathbf{R}^{k'} \to \mathbf{R}^k$ where k' and k are the cardinalities of F' and F respectively. A stochastic process can then be viewed as a family $\{\mu_F\}$ of distributions on various finite dimensional spaces that satisfy the consistency conditions. A theorem of Kolmogorov says that this is not all that different. Any such consistent family arises from a Q on $(\mathbf{X}\,,\mathcal{B})$ which is uniquely determined by the family $\{\mu_F\}$.

If $\mathbf{T}$ is countable this is quite satisfactory. $\mathbf{X}$ is the the space of sequences and the σ-field $\mathcal{B}$ is quite adequate to answer all the questions we may want to ask. The set of bounded sequences, the set of convergent sequences, the set of summable sequences are all measurable subsets of $\mathbf{X}$ and therefore we can answer questions like, does the sequence converge with probability 1, etc. However if $\mathbf{T}$ is uncountable like $[0, T]$, then the space of bounded functions, the space of continuous functions etc, are not measurable sets. They do not belong to $\mathcal{B}$. Basically, in probability theory, the rules involve only a countable collection of sets at one time and any information that involves the values of an uncountable number of measurable functions is out of reach. There is an intrinsic reason for this. In probability theory we can always change the values of a random variable on a set of measure 0 and we have not changed anything of consequence. Since we are allowed to mess up each function on a set of measure 0 we have to assume that each function has indeed been messed up on a set of measure 0. If we are dealing with a countable number of functions the 'mess up' has occured only on the countable union of these invidual sets of measure 0, which by the properties of a measure is again a set of measure 0. On the other hand if we are dealing with an uncountable set of functions, then these sets of measure 0 can possibly gang up on us to produce a set of positive or even full measure. We just can not be sure.

Of course it would be foolish of us to mess things up unnecessarily. If we can clean things up and choose a nice version of our random variables we should do so. But we cannot really do this sensibly unless we decide first what nice means. We however face the risk of being too greedy and it may not be possible to have a version as nice as we seek. But then we can always change our mind.

1.2 Regularity

Very often it is natural to try to find a version that has continuous trajectories. This is equivalent to restricting $\mathbf{X}$ to the space of continuous functions on $[0, T]$ and we are trying to construct a measure Q on $\mathbf{X} = C[0\,,T]$ with the natural σ-field $\mathcal{B}$. This is not always possible. We want to find some sufficient conditions on the finite dimensional distributions $\{\mu_F\}$ that guarantee that a choice of Q exists on $(\mathbf{X}\,,\mathcal{B})$.

Theorem 1.1. (Kolmogorov's Regularity Theorem) *Assume that for any pair $(s,t) \in [0\,,T]$ the bivariate distribution $\mu_{s,t}$ satisfies*

$$\int\int |x-y|^{\beta}\mu_{s,t}(dx\,,dy) \leq C|t-s|^{1+\alpha} \tag{1.2}$$

for some positive constants β, α and C. Then there is a unique Q on $(\mathbf{X}\,,\mathcal{B})$ such that it has $\{\mu_F\}$ for its finite dimensional distributions.

Proof. Since we can only deal effectively with a countable number of random variables, we restrict ourselves to values at diadic times. Let us, for simplicity, take $T=1$. Denote by $\mathbf{T}_n$ time points t of the form $t=\frac{j}{2^n}$ for $0 \leq j \leq 2^n$. The countable union $\cup_{j=0}^{\infty}\mathbf{T}_j = \mathbf{T}^0$ is a countable dense subset of $\mathbf{T}$. We will construct a probability measure Q on the space of sequences corresponding to the values of $\{x(t) : t \in \mathbf{T}^0\}$, show that Q is supported on sequences that produce uniformly continuous functions on $\mathbf{T}^0$ and then extend them automatically to $\mathbf{T}$ by continuity and the extension will provide us the natural Q on $C[0\,,1]$. If we start from the set of values on $\mathbf{T}_n$, the n-th level of diadics, by linear iterpolation we can construct a version $x_n(t)$ that agrees with the original variables at these diadic points. This way we have a sequence $x_n(t)$ such that $x_n(\cdot) = x_{n+1}(\cdot)$ on $\mathbf{T}_n$. If we can show

$$Q\big[x(\cdot) : \sup_{0\leq t\leq 1}|x_n(t) - x_{n+1}(t)| \geq 2^{-n\gamma}\big] \leq C2^{-n\delta} \tag{1.3}$$

then we can conclude that

$$Q\big[x(\cdot) : \lim_{n\to\infty} x_n(t) = x_\infty(t) \ \text{ exists } \ \text{uniformly } \ \text{on } [0\,,1]\big] = 1 \tag{1.4}$$

The limit $x_\infty(\cdot)$ will be continuous on $\mathbf{T}$ and will coincide with $x(\cdot)$ on $\mathbf{T}^0$ there by establishing our result. Proof of (1.3) depends on a simple observation. The difference $|x_n(\cdot) - x_{n+1}(\cdot)|$ achieves its maximum at the mid point of one of the diadic intervals determined by $\mathbf{T}_n$ and hence

$$\begin{aligned}
&\sup_{0\leq t\leq 1}|x_n(t) - x_{n+1}(t)| \\
&\qquad \leq \sup_{1\leq j\leq 2^n}|x_n(\frac{2j-1}{2^{n+1}}) - x_{n+1}(\frac{2j-1}{2^{n+1}})| \\
&\qquad \leq \sup_{1\leq j\leq 2^n}\max\big\{|x(\frac{2j-1}{2^{n+1}}) - x(\frac{2j}{2^{n+1}})|, |x(\frac{2j-1}{2^{n+1}}) - x(\frac{2j-2}{2^{n+1}})|\big\}
\end{aligned}$$

and we can estimate the left hand side of (1.3) by

$$
\begin{aligned}
&Q\big[x(\cdot) : \sup_{0\le t\le 1}|x_n(t)-x_{n+1}(t)|\ge 2^{-n\gamma}\big]\\
&\le Q\big[\sup_{1\le i\le 2^{n+1}}|x(\frac{i}{2^{n+1}})-x(\frac{i-1}{2^{n+1}})|\ge 2^{-n\gamma}\big]\\
&\le 2^{n+1}\sup_{1\le i\le 2^{n+1}} Q\big[|x(\frac{i}{2^{n+1}})-x(\frac{i-1}{2^{n+1}})|\ge 2^{-n\gamma}\big]\\
&\le 2^{n+1}2^{n\beta\gamma}\sup_{1\le i\le 2^{n+1}} E^Q\big[|x(\frac{i}{2^{n+1}})-x(\frac{i-1}{2^{n+1}})|^{\beta}\big]\\
&\le C2^{n+1}\,2^{n\beta\gamma}\,2^{-(1+\alpha)(n+1)}\\
&\le C2^{-n\delta}
\end{aligned}
$$

provided $\delta\le\alpha-\beta\gamma$. For given α,β we can pick $\gamma<\alpha\beta$ and we are done. ☐

An equivalent version of this theorem is the following.

Theorem 1.2. *If $x(t\,,\omega)$ is a stochastic process on $(\Omega\,,\Sigma\,,P)$ satisfying*

$$E^P\left[|x(t)-x(s)|^{\beta}\right]\le C|t-s|^{1+\alpha}$$

for some positive constants α, β and C, then if necessary , $x(t,\omega)$ can be modified for each t on a set of measure zero, to obtain an equivalent version that is almost surely continuous.

As an important application we consider Brownian Motion, which is defined as a stochastic process that has multivariate normal distributions for its finite dimensional distributions. These normal distributions have mean zero and the variance covariance matrix is specified by $Cov(x(s),x(t))=\min(s,t)$. An elementary calculation yields

$$E|x(s)-x(t)|^4=3|t-s|^2$$

so that Theorem 1.1 is applicable with $\beta=4$, $\alpha=1$ and $C=3$.

To see that some restriction is needed, let us consider the Poisson process defined as a process with independent increments with the distribution of $x(t)-x(s)$ being Poisson with parameter $t-s$ provided $t>s$. In this case since

$$P[x(t)-x(s)\ge 1]=1-\exp[-(t-s)]$$

we have, for every $n\ge 0$,

$$E|x(t)-x(s)|^n\ge 1-\exp[-|t-s|]\simeq C|t-s|$$

and the conditions for Theorem 1.1 are never satisfied. It should not be, because after all a Poisson process is a counting process and jumps whenever the event that it is counting occurs and it would indeed be greedy of us to try to put the measure on the space of continuous functions.

Remark 1.1. The fact that there cannot be a measure on the space of continuous functions whose finite dimensional distributions coincide with those of the Poisson process requires a proof. There is a whole class of nasty examples of measures $\{Q\}$ on the space of continuous functions such that for every $t \in [0\,,1]$

$$Q\big[\omega : x(t\,,\omega) \ \text{ is a rational } \ \text{number } \big] = 1$$

The difference is that the rationals are dense, whereas the integers are not. The proof has to depend on the fact that a continuous function that is not identically equal to some fixed integer must spend a positive amount of time at nonintegral points. Try to make a rigorous proof using Fubini's theorem.

1.3 Garsia, Rodemich and Rumsey inequality.

If we have a stochastic process $x(t\,,\omega)$ and we wish to show that it has a nice version, perhaps a continuous one, or even a Holder continuous or differentiable version, there are things we have to estimate. Establishing Holder continuity amounts to estimating

$$\epsilon(\ell) = P\big[\sup_{s,t} \frac{|x(s) - x(t)|}{|t-s|^\alpha} \le \ell\big]$$

and showing that $\epsilon(\ell) \to 1$ as $\ell \to \infty$. These are often difficult to estimate and require special methods. A slight modification of the proof of Theorem 1.1 will establish that the nice, continuous version of Brownian motion actually satisfies a Holder condition of exponent α so long as $0 < \alpha < \frac{1}{2}$.

On the other hand if we want to show only that we have a version $x(t\,,\omega)$ that is square integrable, we have to estimate

$$\epsilon(\ell) = P\big[\int_0^1 |x(t\,,\omega)|^2 dt \le \ell\big]$$

and try to show that $\epsilon(\ell) \to 1$ as $\ell \to \infty$. This task is somewhat easier because we could control it by estimating

$$E^P\big[\int_0^1 |x(t\,,\omega)|^2\, dt\big]$$

and that could be done by the use of Fubini's theorem. After all

$$E^P\big[\int_0^1 |x(t\,,\omega)|^2\, dt\big] = \int_0^1 E^P\big[|x(t\,,\omega)|^2\,\big]\, dt$$

Estimating integrals are easier that estimating suprema. Sobolev inequality controls suprema in terms of integrals. Garsia, Rodemich and Rumsey inequality is a generalization and can be used in a wide variety of contexts.

Theorem 1.3. *Let $\Psi(\cdot)$ and $p(\cdot)$ be continuous strictly increasing functions on $[0\,,\infty)$ with $p(0) = \Psi(0) = 0$ and $\Psi(x) \to \infty$ as $x \to \infty$. Assume that a continuous function $f(\cdot)$ on $[0\,,1]$ satisfies*

$$\int_0^1 \int_0^1 \Psi\left(\frac{|f(t) - f(s)|}{p(|t-s|)}\right) ds\,dt = B < \infty. \tag{1.5}$$

Then

$$|f(0) - f(1)| \le 8 \int_0^1 \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u) \tag{1.6}$$

The double integral (1.5) has a singularity on the diagonal and its finiteness depends on f, p and Ψ. The integral in (1.6) has a singularity at $u = 0$ and its convergence requires a balancing act between $\Psi(\cdot)$ and $p(\cdot)$. The two conditions compete and the existence of a pair $\Psi(\cdot)\,, p(\cdot)$ satisfying all the conditions will turn out to imply some regularity on $f(\cdot)$.

Let us first assume Theorem 1.3 and illustrate its uses with some examples. We will come back to its proof at the end of the section. First we remark that the following corollary is an immediate consequence of Theorem 1.3.

Corollary 1.4. *If we replace the interval $[0\,,1]$ by the interval $[T_1\,, T_2]$ so that*

$$B_{T_1,T_2} = \int_{T_1}^{T_2} \int_{T_1}^{T_2} \Psi\left(\frac{|f(t) - f(s)|}{p(|t-s|)}\right) ds\,dt$$

then

$$|f(T_2) - f(T_1)| \le 8 \int_0^{T_2 - T_1} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u)$$

For $0 \le T_1 < T_2 \le 1$ because $B_{T_1,T_2} \le B_{0,1} = B$, we can conclude from (1.5), that the modulus of continuity $\varpi_f(\delta)$ satisfies

$$\varpi_f(\delta) = \sup_{\substack{0 \le s,t \le 1 \\ |t-s| \le \delta}} |f(t) - f(s)| \le 8 \int_0^{\delta} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u) \tag{1.7}$$

Proof. (of Corollary). If we map the interval $[T_1\,, T_2]$ into $[0\,,1]$ by $t' = \frac{t - T_1}{T_2 - T_1}$ and redefine $f'(t) = f(T_1 + (T_2 - T_1)t)$ and $p'(u) = p((T_2 - T_1)u)$, then

$$\begin{aligned}
&\int_0^1 \int_0^1 \Psi\Big[\frac{|f'(t) - f'(s)|}{p'(|t-s|)}\Big] ds\,dt \\
&\quad = \frac{1}{(T_2 - T_1)^2} \int_{T_1}^{T_2} \int_{T_1}^{T_2} \Psi\Big[\frac{|f(t) - f(s)|}{p(|t-s|)}\Big] ds\,dt \\
&\quad = \frac{B_{T_1,T_2}}{(T_2 - T_1)^2}
\end{aligned}$$

and

$$
\begin{aligned}
|f(T_2) - f(T_1)| &= |f'(1) - f'(0)| \\
&\le 8 \int_0^1 \Psi^{-1}\left(\frac{4B_{T_1,T_2}}{(T_2 - T_1)^2\, u^2} \right) dp'(u) \\
&= 8 \int_0^{(T_2 - T_1)} \Psi^{-1}\left(\frac{4B_{T_1,T_2}}{u^2} \right) dp(u)
\end{aligned}
$$

In particular (1.7) is now an immediate consequence. □

Let us now turn to Brownian motion or more generally processes that satisfy

$$
E^P\Big[|x(t) - x(s)|^\beta \Big] \le C|t - s|^{1+\alpha}
$$

on $[0\,,1]$. We know from Theorem 1.1 that the paths can be chosen to be continuous. We will now show that the continuous version enjoys some additional regularity. We apply Theorem 1.3 with $\Psi(x) = x^\beta$, and $p(u) = u^{\frac{\gamma}{\beta}}$. Then

$$
\begin{aligned}
&E^P\left[\int_0^1 \int_0^1 \Psi\left(\frac{|x(t) - x(s)|}{p(|t - s|)} \right) ds\, dt \right] \\
&= \int_0^1 \int_0^1 E^P\left[\frac{|x(t) - x(s)|^\beta}{|t - s|^\gamma} \right] dsdt \\
&\le C \int_0^1 \int_0^1 |t - s|^{1+\alpha-\gamma}\, dsdt \\
&= C\, C_\delta
\end{aligned}
$$

where C_δ is a constant depending only on $\delta = 2 + \alpha - \gamma$ and is finite if $\delta > 0$. By Fubini's theorem, almost surely

$$
\int_0^1 \int_0^1 \Psi\left(\frac{|x(t) - x(s)|}{p(|t - s|)} \right) ds\, dt = B(\omega) < \infty
$$

and by Tchebychev's inequality

$$
P\big[B(\omega) \ge B\big] \le \frac{C\, C_\delta}{B}.
$$

On the other hand

$$
\begin{aligned}
8 \int_0^h \left(\frac{4B}{u^2}\right)^{\frac{1}{\beta}} du^{\frac{\gamma}{\beta}} &= 8 \frac{\gamma}{\beta} (4B)^{\frac{1}{\beta}} \int_0^h u^{\frac{\gamma-2}{\beta} - 1} du \\
&= 8 \frac{\gamma}{\gamma - 2} (4B)^{\frac{1}{\beta}} h^{\frac{\gamma-2}{\beta}}
\end{aligned}
$$

We obtain Holder continuity with exponent $\frac{\gamma-2}{\beta}$ which can be anything less than $\frac{\alpha}{\beta}$. For Brownian motion $\alpha = \frac{\beta}{2} - 1$ and therefore $\frac{\alpha}{\beta}$ can be made arbitrarily close to $\frac{1}{2}$.

Remark 1.2. With probability 1 Brownian paths satisfy a Holder condition with any exponent less than $\frac{1}{2}$.

It is not hard to see that they do not satisfy a Holder condition with exponent $\frac{1}{2}$

Exercise 1.1. Show that

$$P\Big[\sup_{0\le s,\,t\le 1}\frac{|x(t)-x(s)|}{\sqrt{|t-s|}}=\infty\Big]=1.$$

Hint: The random variables $\frac{x(t)-x(s)}{\sqrt{|t-s|}}$ have standard normal distributions for any interval $[s,t]$ and they are independent for disjoint intervals. We can find as many disjoint intervals as we wish and therefore dominate the Holder constant from below by the supremum of absolute values of an arbitrary number of independent Gaussians.

Exercise 1.2. (Precise modulus of continuity). The choice of $\Psi(x)=\exp[\alpha\,x^2]$ with $\alpha<\frac{1}{2}$ and $p(u)=u^{\frac{1}{2}}$ produces a modulus of continuity of the form

$$\varpi_x(\delta)\le 8\int_0^\delta\sqrt{\frac{1}{\alpha}\log\big[1+\frac{4B}{u^2}\big]}\frac{1}{2\sqrt{u}}du$$

that produces eventually a statement

$$P\Big[\limsup_{\delta\to 0}\frac{\varpi_x(\delta)}{\sqrt{\delta\log\frac{1}{\delta}}}\le 16\Big]=1.$$

Remark 1.3. This is almost the final word, because the argument of the previous exercise can be tightened slightly to yield

$$P\Big[\limsup_{\delta\to 0}\frac{\varpi_x(\delta)}{\sqrt{\delta\log\frac{1}{\delta}}}\ge\sqrt{2}\Big]=1$$

and according to a result of Paul Lévy

$$P\Big[\limsup_{\delta\to 0}\frac{\varpi_x(\delta)}{\sqrt{\delta\log\frac{1}{\delta}}}=\sqrt{2}\Big]=1.$$

Proof. (of Theorem 1.3.) Define

$$I(t)=\int_0^1\Psi\left(\frac{|f(t)-f(s)|}{p(|t-s|)}\right)ds$$

and

$$B=\int_0^1 I(t)\,dt$$

There exists $t_0 \in (0\,,1)$ such that $I(t_0) \le B$. We shall prove that

$$|f(0) - f(t_0)| \le 4\int_0^1 \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u) \tag{1.8}$$

By a similar argument

$$|f(1) - f(t_0)| \le 4\int_0^1 \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u)$$

and combining the two we will have (1.6). To prove 1.8 we shall pick recursively two sequences $\{u_n\}$ and $\{t_n\}$ satisfying

$$t_0 > u_1 > t_1 > u_2 > t_2 > \cdots > u_n > t_n > \cdots$$

in the following manner. By induction, if t_{n-1} has already been chosen, define

$$d_n = p(t_{n-1})$$

and pick u_n so that $p(u_n) = \frac{d_n}{2}$. Then

$$\int_0^{u_n} I(t)\,dt \le B$$

and

$$\int_0^{u_n} \Psi\left(\frac{|f(t_{n-1}) - f(s)|}{p(|t_{n-1} - s|)}\right) ds \le I(t_{n-1})$$

Now t_n is chosen so that

$$I(t_n) \le \frac{2B}{u_n}$$

and

$$\Psi\left(\frac{|f(t_n) - f(t_{n-1})|}{p(|t_n - t_{n-1}|)}\right) \le 2\frac{I(t_{n-1})}{u_n} \le \frac{4B}{u_{n-1}\,u_n} \le \frac{4B}{u_n^2}$$

We now have

$$|f(t_n) - f(t_{n-1})| \le \Psi^{-1}\left(\frac{4B}{u_n^2}\right) p(t_{n-1} - t_n) \le \Psi^{-1}\left(\frac{4B}{u_n^2}\right) p(t_{n-1}).$$

$$p(t_{n-1}) = 2p(u_n) = 4[p(u_n) - \frac{1}{2}p(u_n)] \le 4[p(u_n) - p(u_{n+1})]$$

Then,

$$|f(t_n) - f(t_{n-1})| \le 4\Psi^{-1}\left(\frac{4B}{u_n^2}\right)[p(u_n) - p(u_{n+1})] \le 4\int_{u_{n+1}}^{u_n} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u)$$

Summing over $n = 1, 2, \cdots$, we get

$$|f(t_0) - f(0)| \le 4\int_0^{u_1} \Psi^{-1}\left(\frac{4B}{u^2}\right) p(du) \le 4\int_0^{u_1} \Psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

and we are done. $\square$

Example 1.1. Let us consider a stationary Gaussian process with

$$\rho(t) = E[X(s)X(s+t)]$$

and denote by

$$\sigma^2(t) = E[(X(t) - X(0))^2] = 2(\rho(0) - \rho(t)).$$

Let us suppose that $\sigma^2(t) \le C|\log t|^{-a}$ for some $a > 1$ and $C < \infty$. Then we can apply Theorem 1.3 and establish the existence of an almost sure continuous version by a suitable choice of Ψ and p.

On the other hand we will show that, if $\sigma^2(t) \ge c|\log t|^{-1}$, then the paths are almost surely unbounded on every time interval. It is generally hard to prove that some thing is unbounded. But there is a nice trick that we will use. One way to make sure that a function $f(t)$ on $t_1 \le t \le t_2$ is unbounded is to make sure that the measure $\mu_f(A) = \text{LebMes}\ \{t : f(t) \in A\}$ is not supported on a compact interval. That can be assured if we show that μ_f has a density with respect to the Lebsgue measure on $\mathbf{R}$ with a density $\phi_f(x)$ that is real analytic, which in turn will be assured if we show that

$$\int_{-\infty}^{\infty} |\widehat{\mu}_f(\xi)| e^{\alpha|\xi|}\, d\xi < \infty$$

for some $\alpha > 0$. By Schwarz's inequality it is sufficient to prove that

$$\int_{-\infty}^{\infty} |\widehat{\mu}_f(\xi)|^2 e^{\alpha|\xi|}\, d\xi < \infty$$

for some $\alpha > 0$. We will prove

$$\int_{-\infty}^{\infty} E\left[\left|\int_{t_1}^{t_2} e^{i\,\xi\, X(t)}\, dt\right|^2\right] e^{\alpha\xi}\, d\xi < \infty$$

for some $\alpha > 0$. Sine we can replace α by $-\alpha$, this will control

$$\int_{-\infty}^{\infty} E\left[\left|\int_{t_1}^{t_2} e^{i\,\xi\, X(t)}\, dt\right|^2\right] e^{\alpha|\xi|}\, d\xi < \infty$$

and we can apply Fubini's theorem to complete the proof.

$$
\begin{aligned}
\int_{-\infty}^{\infty} & E\left[\left|\int_{t_1}^{t_2} e^{i\,\xi\,X(t)}\,dt\right|^2\right] e^{\alpha\xi}\,d\xi \\
&= \int_{-\infty}^{\infty} E\left[\int_{t_1}^{t_2}\int_{t_1}^{t_2} e^{i\,\xi\,(X(t)-X(s))}\,ds\,dt\right] e^{\alpha\xi}\,d\xi \\
&= \int_{-\infty}^{\infty}\int_{t_1}^{t_2}\int_{t_1}^{t_2} E\left[e^{i\,\xi\,(X(t)-X(s))}\right] ds\,dt\,e^{\alpha\xi}\,d\xi \\
&= \int_{-\infty}^{\infty}\int_{t_1}^{t_2}\int_{t_1}^{t_2} e^{-\frac{\sigma^2(t-s)\xi^2}{2}}\,ds\,dt\,e^{\alpha\xi}\,d\xi \\
&= \int_{t_1}^{t_2}\int_{t_1}^{t_2} \frac{\sqrt{2\pi}}{\sigma(t-s)} e^{\frac{\alpha^2}{2\sigma^2(t-s)}} \\
&\le \int_{t_1}^{t_2}\int_{t_1}^{t_2} \frac{\sqrt{2\pi}}{\sigma(t-s)} e^{\frac{\alpha^2|\log|(t-s)||}{2c}}\,ds\,dt \\
&< \infty
\end{aligned}
$$

provided α is small enough.

1.4 Brownian Motion as a Martingale

P is the Wiener measure on $(\Omega, \mathcal{B})$ where $\Omega = C[0,T]$ and $\mathcal{B}$ is the Borel σ-field on Ω. In addition we denote by $\mathcal{B}_t$ the σ-field generated by $x(s)$ for $0 \le s \le t$. It is easy to see tha $x(t)$ is a martingale with respect to $(\Omega, \mathcal{B}_t, P)$, i.e for each $t > s$ in $[0,T]$

$$E^P\{x(t)|\mathcal{B}_s\} = x(s) \quad \text{a.e.} \quad P \tag{1.9}$$

and so is $x(t)^2 - t$. In other words

$$E^P\{x(t)^2 - t\,|\mathcal{F}_s\} = x(s)^2 - s \quad \text{a.e.} \quad P \tag{1.10}$$

The proof is rather straight forward. We write $x(t) = x(s) + Z$ where $Z = x(t) - x(s)$ is a random variable independent of the past history $\mathcal{B}_s$ and is distributed as a Gaussian random variable with mean 0 and variance $t - s$. Therefore $E^P\{Z|\mathcal{B}_s\} = 0$ and $E^P\{Z^2|\mathcal{B}_s\} = t - s$ a.e P. Conversely,

Theorem 1.5. Lévy's theorem. *If P is a measure on $(C[0,T], \mathcal{B})$ such that $P[x(0) = 0] = 1$ and the the functions $x(t)$ and $x^2(t) - t$ are martingales with respect to $(C[0,T], \mathcal{B}_t, P)$ then P is the Wiener measure.*

Proof. The proof is based on the observation that a Gaussian distribution is determined by two moments. But that the distribution is Gaussian is a consequence of the fact that the paths are almost surely continuous and not part of our assumptions. The actual proof is carried out by establishing that for each real number λ

$$X_\lambda(t) = \exp\big[\lambda x(t) - \frac{\lambda^2}{2}t\big] \tag{1.11}$$

is a martingale with respect to $(C[0,T], \mathcal{B}_t, P)$. Once this is established it is elementary to compute

$$E^P\big[\exp\big[\lambda(x(t)-x(s))\big]|\mathcal{B}_s\big] = \exp\big[\frac{\lambda^2}{2}(t-s)\big]$$

which shows that we have a Gaussian Process with independent increments with two matching moments. The proof of (1.11) is more or less the same as proving the central limit theorem. In order to prove (2.5) we can assume with out loss of generality that $s = 0$ and will show that

$$E^P\big[\exp\big[\lambda x(t) - \frac{\lambda^2}{2}t\big]\big] = 1 \tag{1.12}$$

To this end let us define successively $\tau_{0,\epsilon} = 0$,

$$\tau_{k+1,\epsilon} = \min\big[\inf\big\{s : s \geq \tau_{k,\epsilon}, |x(s) - x(\tau_{k,\epsilon})| \geq \epsilon\big\}, t\,, \tau_{k,\epsilon} + \epsilon\big]$$

Then each $\tau_{k,\epsilon}$ is a stopping time and eventually $\tau_{k,\epsilon} = t$ by continuity of paths. The continuity of paths also guarantees that $|x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})| \leq \epsilon$. We write

$$x(t) = \sum_{k\geq 0}[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})]$$

and

$$t = \sum_{k\geq 0}[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]$$

To establish (1.12) we calculate the quantity on the left hand side as

$$\lim_{n\to\infty} E^P\big[\exp\big[\sum_{0\leq k\leq n}\big[\lambda[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - \frac{\lambda^2}{2}[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\big]\big]$$

and show that it is equal to 1. Let us cosider the σ-field $\mathcal{F}_k = \mathcal{B}_{\tau_{k,\epsilon}}$ and the quantity

$$q_k(\omega) = E^P\big[\exp\big[\lambda[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - (\frac{\lambda^2}{2} + \delta)[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\Big|\mathcal{F}_k\big]$$

where $\delta = \delta(\epsilon, \lambda)$ is to be chosen later such that $0 \leq \delta(\epsilon,\lambda) \leq 1$ and $\delta(\epsilon,\lambda) \to 0$ as $\epsilon \to 0$ for every fixed λ.

If z and τ are random variables bounded by $\epsilon > 0$ such that

$$E[z] = E[z^2 - \tau] = 0$$

then for any $0 \leq \delta \leq 1$,

$$\begin{aligned}
&E\big[\exp[\lambda z - (\frac{\lambda^2}{2} + \delta)\tau]\big] \\
&\qquad \leq E\big[1 + [\lambda z - (\frac{\lambda^2}{2} + \delta)\tau] + \frac{1}{2}[\lambda z - (\frac{\lambda^2}{2} + \delta)\tau]^2 + C_\lambda[|z|^3 + \tau^3]\big] \\
&\qquad \leq E\big[1 - \delta\tau + C_\lambda \epsilon\tau\big] \\
&\qquad \leq 1
\end{aligned}$$

provided $\delta \geq C_\lambda\epsilon$. Clearly there is a choice of $\delta = \delta(\epsilon, \lambda) \to 0$ as $\epsilon \to 0$ such that $q_k(\omega) \leq 1$ for every k and almost all ω. In particular, by induction

$$E^P\big[\exp\big[\sum_{0\leq k\leq n}\big[\lambda[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - (\frac{\lambda^2}{2} + \delta)[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\big]\big] \leq 1$$

for every n, and by Fatou's lemma

$$E^P\big[\exp\big[\lambda(x(t) - x(0)) - (\frac{\lambda^2}{2} + \delta)t\big]\big] \leq 1.$$

Since $\epsilon > 0$ is arbitrary we have proved one half of (1.12). The argument actually proves that

$$E^P\big[\exp\big[\lambda(x(t) - x(s)) - \frac{\lambda^2}{2}(t-s)\big]|\mathcal{F}_s\big] \leq 1 \qquad \text{a.e.}$$

To prove the other half, we t note that $Z_\lambda(t) = \exp\big[\lambda(x(t) - x(s)) - \frac{\lambda^2}{2}(t-s)\big]$ is a supermartingale and from Doob's inequality we can get a tail estiamte

$$P[\sup_{0\leq s\leq t}|x(s) - x(0)| \geq \ell] \leq 2\exp[-\frac{\ell^2}{2t}]$$

Since this allows us to use the dominated converegence theorem, it will be sufficient to establish

$$E^P\big[\exp\big[\sum_{0\leq k\leq n}\big[\lambda[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - (\frac{\lambda^2}{2} - \delta)[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\big]\big] \geq 1$$

which follows by induction from

$$\begin{aligned}
&E\big[\exp[\lambda z - (\frac{\lambda^2}{2} - \delta)\tau]\big] \\
&\qquad \geq E\big[1 + [\lambda z - (\frac{\lambda^2}{2} - \delta)\tau] + \frac{1}{2}[\lambda z - (\frac{\lambda^2}{2} - \delta)\tau]^2 - C_\lambda[|z|^3 + \tau^3]\big] \\
&\qquad \leq E\big[1 + \delta\tau - C_\lambda \epsilon\tau\big] \\
&\qquad \geq 1
\end{aligned}$$

This completes the proof of the theorem. $\square$

Exercise 1.3. Why does Theorem 1.5 fail for the process $x(t) = N(t) - t$ where $N(t)$ is the standard Poisson Process with rate 1?

Remark 1.4. One can use the Martingale inequality in order to estimate the probability $P\{\sup_{0\le s\le t} |x(s)| \ge \ell\}$. For $\lambda > 0$, by Doob's inequality

$$P\big[\sup_{0\le s\le t} \exp\big[\lambda x(s) - \frac{\lambda^2}{2}s\big] \ge A\big] \le \frac{1}{A}$$

and

$$\begin{aligned} P\big[\sup_{0\le s\le t} x(s) \ge \ell\big] &\le P\big[\sup_{0\le s\le t} [x(s) - \frac{\lambda s}{2}] \ge \ell - \frac{\lambda t}{2}\big] \\ &= P\big[\sup_{0\le s\le t} [\lambda x(s) - \frac{\lambda^2 s}{2}] \ge \lambda\ell - \lambda^2 t 2\big] \\ &\le \exp[-\lambda\ell + \frac{\lambda^2 t}{2}] \end{aligned}$$

Optimizing over $\lambda > 0$, we obtain

$$P\big[\sup_{0\le s\le t} x(s) \ge \ell\big] \le \exp[-\frac{\ell^2}{2t}]$$

and by symmetry

$$P\big[\sup_{0\le s\le t} |x(s)| \ge \ell\big] \le 2\exp[-\frac{\ell^2}{2t}]$$

The estimate is not too bad because by reflection principle

$$P\big[\sup_{0\le s\le t} x(s) \ge \ell\big] = 2\,P\big[x(t) \ge \ell\big] = \sqrt{\frac{2}{\pi\, t}}\int_\ell^\infty \exp[-\frac{x^2}{2\,t}]\,dx$$

Exercise 1.4. One can use the estimate above to prove the result of Paul Lévy

$$P\Big[\limsup_{\delta\to 0} \frac{\sup_{\substack{0\le s,t\le 1\\ |s-t|\le\delta}} |x(s) - x(t)|}{\sqrt{\delta\log\frac{1}{\delta}}} = \sqrt{2}\Big] = 1$$

We had an exercise in the previous section that established the lower bound. Let us concentrate on the upper bound. If we define

$$\Delta_\delta(\omega) = \sup_{\substack{0\le s,t\le 1\\ |s-t|\le\delta}} |x(s) - x(t)|$$

first check that it is sufficient to prove that for any $\rho < 1$, and $a > \sqrt{2}$

$$\sum_n P\big[\Delta_{\rho^n}(\omega) \ge a\sqrt{n\rho^n \log\frac{1}{\rho}}\big] < \infty \tag{1.13}$$

To estimate $\Delta_{\rho^n}(\omega)$ it is sufficient to estimate $\sup_{t\in I_j}|x(t)-x(t_j)|$ for $k_\epsilon\rho^{-n}$ overlapping intervals $\{I_j\}$ of the form $[t_j, t_j+(1+\epsilon)\rho^n\,]$ with length $(1+\epsilon)\rho^n$. For each $\epsilon>0$, $k_\epsilon=\epsilon^{-1}$ is a constant such that any interval $[s\,,t]$ of length no larger than ρ^n is completely contained in some I_j with $t_j\le s\le t_j+\epsilon\rho^n$. Then

$$\Delta_{\rho^n}(\omega)\le \sup_j\big[\,\sup_{t\in I_j}|x(t)-x(t_j)|+\sup_{t_j\le s\le t_j+\epsilon\rho^n}|x(s)-x(t_j)|\big]$$

Therefore, for any $a=a_1+a_2$,

$$\begin{aligned}
&P\Big[\Delta_{\rho^n}(\omega)\ge a\sqrt{n\rho^n\log\frac{1}{\rho}}\,\Big]\\
&\qquad\le P\Big[\sup_j\sup_{t\in I_j}|x(t)-x(t_j)|\ge a_1\sqrt{n\rho^n\log\frac{1}{\rho}}\Big]\\
&\qquad\quad+P\Big[\sup_j\sup_{t_j\le s\le t_j+\epsilon\rho^n}|x(s)-x(t_j)|\ge a_2\sqrt{n\rho^n\log\frac{1}{\rho}}\Big]\\
&\qquad\le 2k_\epsilon\rho^{-n}\Big[\exp[-\frac{a_1^2\,n\rho^n\log\frac{1}{\rho}}{2(1+\epsilon)\rho^n}]+\exp[-\frac{a_2^2\,n\rho^n\log\frac{1}{\rho}}{2\epsilon\rho^n}]\Big]
\end{aligned}$$

Since $a>\sqrt{2}$, we can pick $a_1>\sqrt{2}$ and $a_2>0$. For $\epsilon>0$ sufficiently small (1.13) is easily verified.