Problemset 3.

Let $x(t)$ be stochastic process on $(\Omega, \mathcal{F}_t, P)$ such that $x(t)$ is progressively measurable and almost surely continuous. Moreover let

$$y(t) = x(t) - x(0) - \int_0^t b(s, x(s))ds$$

and

$$y^2(t) - \int_0^t a(s, x(s))ds$$

be martingales with respect to $(\Omega, \mathcal{F}_t, P)$. Here $b(t, x)$ and $a(t, x)$ are bounded measurable coefficients, with $a(t, x) \geq 0$. Show that for smooth, bounded f,

$$f(x(t)) - f(x(0)) - \int_0^t [\frac{1}{2}a(s, x(s))f''(x(s)) + b(s, x(s))f'(x(s))]ds$$

is a Martingale with respect to $(\Omega, \mathcal{F}_t, P)$ and so is

$$f(y(t)) - f(y(0)) - \int_0^t [\frac{1}{2}a(s, x(s))f''(y(s))]ds.$$