Let $C(X)$ be the space of continuous functions on a compact metric space X. Let $\Lambda(f)$ be a non-negative linear functional on $C(X)$. Then there is a measure μ on the Borel sets $\mathcal{B}$ of X such that

$$\Lambda(f) = \int f \, d\mu$$

for all $f \in C(X)$.

We define for closed sets $C \in X$

$$\alpha(C) = \inf \big[\Lambda(f),\ f : f \geq \mathbf{1}_C\big],$$

and

$$\beta(G) = \sup \big[\alpha(C),\ C : C \subset G\}\big]$$

for open sets G. Finally we define

$$\mu_*(A) = \sup\ \big[\alpha(C),\ C : C \subset A\big] \leq \inf\ \big[\beta(G),\ G : G \supset A\big] = \mu^*(A)$$

and show that on $\mathcal{B}$ $\mu^*(A) = \mu_*(A)$ and $\mu(A)$ is a countably additive measure on $\mathcal{B}$.

Step 1.

$$\alpha(C_1 \cup C_2) \leq \alpha(C_1) + \alpha(C_2)$$

If $C_1 \cap C_2 = \emptyset$, then

$$\alpha(C_1 \cup C_2) = \alpha(C_1) + \alpha(C_2)$$

Proof: If, for $i = 1, 2$, f_i are chosen such that $\lambda(f_i) \leq \alpha(C_i) + \epsilon$, then $f = f_1 + f_2 \geq \mathbf{1}_{C_1 \cup C_2}$ and $\lambda(f) \leq \alpha(C_1) + \alpha(C_2) + 2\epsilon$. This implies

$$\alpha(C_1 \cup C_2) \leq \Lambda(f) \leq \alpha(C_1) + \alpha(C_2) + 2\epsilon.$$

On the other hand if C_1 and C_2 are disjoint then we can find ϕ with $0 \leq \phi \leq 1$ and $\phi = 1$ on C_1 and 0 on C_2. If f is chosen such that $f \geq \mathbf{1}_{C_1 \cup C_2}$ and

$$\Lambda(f) \leq \alpha(C_1 \cup C_2) + \epsilon$$

then $f_1 = \phi f \geq \mathbf{1}_{C_1}$ and $f_2 = (1 - \phi) f \geq \mathbf{1}_{C_2}$. We have $\lambda(f_i) \geq \alpha(C_i)$ and $f = f_1 + f_2$. This means

$$\alpha(C_1 \cup C_2) + \epsilon \geq \alpha(C_1) + \alpha(C_2)$$

Since $\epsilon > 0$ is arbitrary we are done.

Step 2. $\beta(G_1 \cup G_2) \leq \beta(G_1) + \beta(G_2)$ with equality for disjoint open sets. If $G = \cup_i G_i$ is a countable union of disjoint open sets, then

$$\beta(G) = \sum_i \beta(G_i)$$

Proof: Let $G = G_1 \cup G_2$ and $C \subset G$. Then

$$C_1 = \{x : x \in C \& d(x, G_1^c) \geq d(x, G_2^c)\}$$

and

$$C_2 = \{x : x \in C \& d(x, G_1^c) \leq d(x, G_2^c)\}$$

define sets $C_i \in G_i$. In particular if $\alpha(C) \geq \beta(G) - \epsilon$,

$$\beta(G) \leq \alpha(C) + \epsilon \leq \alpha(C_1) + \alpha(C_2) + \epsilon = \beta(G_1) + \beta(G_2) + \epsilon$$

and we are done. If G_1 and G_2 are disjoint and $C_i \subset G_1$, with $\alpha(C_i) \simeq \beta(G_i)$, then $\beta(G) \geq \alpha(C_1 \cup C_2) = \alpha(C_1) + \alpha(C_2) \simeq \beta(G_1) + \beta(G_2)$.

Clearly for countable union G of disjoint open sets sets G_j,

$$\beta(G) \geq \beta(\cup_{j=1}^n G_j) = \sum_{j=1}^n \beta(G_j)$$

for every n. We need to show that for any union

$$\beta(G) \leq \sum_j \beta(G_j)$$

Let $C \subset G$ be such that $\beta(G) \leq \alpha(C) + \epsilon$. $\cup_j G_j$ covers C which is compact, and will be covered by $\cup_{j=1}^n G_j$ for some n. Then

$$\beta(G) \leq \alpha(C) + \epsilon \leq \sum_{j=1}^n \beta(G_j) + \epsilon \leq \sum_j \beta(G_j) + \epsilon$$

Step 3. If $C \subset G$, $G \cap C^c$ is open and $\beta(G \cap C^c) = \beta(G) - \alpha(C)$.

Proof: First note that if $B \in G \cap C^c$ is closed then C, B are disjoint closed sets, with $B \cup C \subset G$. Therefore $\beta(G) \geq \alpha(C \cup B) = \alpha(C) + \alpha(B)$. This is true for all $B \subset G \cup C^c$. Hence

$$\beta(G) \geq \alpha(C) + \beta(G \cap C^c)$$

On the other hand, for a given ϵ, if f is chosen such that, $0 \leq f \leq 1$, $f \geq \mathbf{1}_C$ and $\Lambda(f) \leq \alpha(C) + \epsilon$ then $f \geq (1+\epsilon)^{-1}\mathbf{1}_U$ for some open set U containing C and if $B \subset U$ we have $\Lambda(f) \geq (1+\epsilon)^{-1}\alpha(B)$. This yields $\beta(U) \leq \sup_B \alpha(B) \leq (1+\epsilon)\Lambda(f) \leq \Lambda(f) + \epsilon < \alpha(C) + 2\epsilon$

$$\beta(G) \leq \beta(U) + \beta(G \cap C^c) \leq \alpha(C) + \beta(G \cap C^c) + \epsilon$$

Step 4. We now show that $\mathcal{A} = \{A : \sup_{C:C\subset A} \alpha(A) = \inf_{G:A\subset G} \beta(G)\}$, is a σ-field and

$$\mu(A) = \sup_{C:C\subset A} \alpha(A) = \inf_{G:A\subset G} \beta(G)$$

is a countably additive measure on A. To show that $A \in \mathcal{A}$, all we need is to find for given ϵ a closed set C and an open set G such that $C \subset A \subset G$ and

$$\beta(G \cap C^c) = \beta(G) - \alpha(C) \leq \epsilon$$

The condition is symmetric in A and A^c because $(G^c)^c = G$. Clearly if

$$\beta(G_i \cap C_i^c) = \beta(G_i) - \alpha(C_i) \leq \epsilon_i$$

works for A_i then $G = \cup_{i=1}^N G_i$ and $C = \cup_{i=1}^N C_i$ works for $\cup_{i=1}^N A_i$ with $\epsilon = \sum_i \epsilon_i$. $\mathcal{A}$ is a therefore a field. If $\{A_i\}$ is a finite collection of disjoint sets in $\mathcal{A}$, from the additivity of $\alpha(C)$ for finite disjoint collection sets $\{C_i\}$ it follows that if $A_i \in \mathcal{A}$ and $\{A_i\}$ are disjoint

$$\mu(\cup_{i=1}^N A_i) = \sum_{i=1}^N \mu(A_i)$$

To show that it is a σ-field, let A_i be a countable disjoint sequence of sets in $\mathcal{A}$. Clearly $\sum_{i=1}^N \mu(A_i) = \mu(\cup_{i=1}^N A_i) \leq \Lambda(1) < \infty$ Therefore for some finite N

$$\sum_{i=1}^N \mu(A_i) \geq \sum_{i=1}^\infty \mu(A_i) - \epsilon$$

Now if we construct C_i, G_i such that $C_i \subset A_i \subset G_i$

$$\alpha(C_i) \leq \mu(A_i) \leq \beta(G_i)$$

and $\beta(G_i) - \alpha(C_i) \leq \frac{\epsilon}{2^i}$, it is easy to see that

$$\cup_{i=1}^N C_i \subset \cup_{i=1}^\infty A_i \leq \cup_{i=1}^\infty G_i$$

and

$$\beta(\cup_{i=1}^\infty G_i) - \alpha(\cup_{i=1}^N C_i) \leq 2\epsilon$$

proving that $\mathcal{A}$ is a σ-field and μ is countably additive on it.

Step 5. Finally we need to prove, that $\lambda(f) = \int f \, d\mu$. Can assume that $0 \leq f \leq 1$. Enough to prove $\lambda(f) \geq \int f \, d\mu$, because if at the same time $\Lambda(1-f) \geq \int (1-f) d\mu$ and $\lambda(1) = \mu(X)$ we have equality.

Let $[a_i, b_i]$ be a finite collection of disjoint subintervals in $[0,1]$ with $b_i - a_i \leq \epsilon$ abd $\mu[\{x : f(x) \not\in n \cup [a_i, b_i]\}] \leq \epsilon$. Let $C_i = \{x : f(x) \in [a_i, b_i]\}$. Construct functions ϕ_i such that $\phi_i = 1$ on C_i satisfies $0 \leq \phi_i \leq 1$ and is 0 on every other C_j. Define $\psi_1 = \phi_1$ and $\psi_j = (1-\phi_1)\cdots(1-\phi_{j-1})\phi_j$ for $2 \leq j \leq n-1$ and $\psi_n = (1-\phi_1)\cdots(1-\phi_{n-1})$. Then $\sum_{j=1}^n \psi_j = 1$ and $\psi_j = 1$ on C_j.

Then $f = \sum_{j=1}^n f\psi_j = \sum_j f_j$ and $f_j = f\psi_j \geq a_j \mathbf{1}_{C_i}$. Therefore

$$\int f \, d\mu \leq \sum_i b_i \mu(C_i) + \epsilon \leq \sum_i b_i \mu(C_i) + 2\epsilon \leq \sum_i \Lambda(f_i) + 2\epsilon \leq \Lambda(f) + 2\epsilon$$

Hence $\lambda(f) \geq \int f \, d\mu$