Assignment 8.

$$F(x) = \sum_{j: x_j \le x} p_j$$

$$A_q = \{x : \limsup_{h \to 0} \frac{F(x+h) - F(x)}{h} > q\}$$

We can exclude from A the set $\{x_j\}$ which is only countable. For each $x \in A_q$ given any $\delta > 0$, there exists $h < \delta$ such that

$$F(x+h) - F(x) \ge qh$$

Since $F(x)$ is right continuous, one can assume that $x+h$ as well is not one of the discontinuity points $\{x_j\}$. The intervals $[x, x+h]$ form a covering of A_q. We can extract a Vitali sub-cover. In other words, given $\epsilon > 0$, we have intervals $\{[x_i, x_i + h_i]\}$ that are disjoint $F(x_i + h_i) - F(x_i) \ge qh_i$ and $\sum_i h_i \ge (\mu(A_q) - \epsilon)$. This implies that

$$q(\mu(A_q) - \epsilon) \le \sum_i [F(x_i + h_i) - F(x_i)] \le s$$

Since $\epsilon > 0$ is arbitrary,

$$\mu(A_q) \le \frac{s}{q}$$

At the second step, given $\eta > 0$ we pick N such that

$$\sum_{j=N+1}^{\infty} p_j \le \eta$$

remove the big jumps and write $F = F_1 + F_2$ where

$$F_1(x) = \sum_{\substack{j \ge N+1 \\ x_j \le x}} p_j$$

and

$$F_2(x) = \sum_{\substack{j \le N \\ x_j \le x}} p_j$$

For F_1 with many small jumps that add up to at most η

$$\mu(B_q) \le \frac{\eta}{q}$$

where

$$B_q = \{x : \limsup_{h \to 0} \frac{F_1(x+h) - F_1(x)}{h} > q\}$$

As for F_2 which has only finitely many jumps, for any x which is not one of the jump points $\{x_1, \ldots, x_N\}$, $F_2(x+h) = F_2(x)$ if h is so small that $[x, x+h]$ has none of these points. Therefore

$$\lim_{h\to 0} \frac{F_2(x+h) - F_2(x)}{h} = 0$$

Hence $A_q \subset B_q \cup \{x_1, x_2, \ldots, x_N\}$ and

$$\mu(A_q) \leq \mu(B_q) \leq \frac{\eta}{\epsilon}$$

Since η can be made as small as we like, for any $q > 0$,

$$\mu(A_q) = 0$$

Assignment 7.

Problem 1. Assume that $\{f_n\}$ is NOT uniformly integrable. Then there exists a subsequence n_j and measurable subsets A_{n_j} of X, such that $\mu(A_{n_j}) \to 0$ while

$$\int_{A_{n_j}} f_{n_j}(x) d\mu \geq \delta > 0$$

Lets us denote A_{n_j} by B_j and f_{n_j} by g_j. $g_j \to f$ in measure. Since $\mu(B_j) \to 0$, it follows that $g_j \mathbf{1}_{B_j^c} \to f$ in measure as well. From Fatou's lemma

$$\int f d\mu \leq \liminf_{j\to\infty} \int g_j \mathbf{1}_{B_j^c} d\mu = \liminf_{j\to\infty} \int [g_j - g_j \mathbf{1}_{B_j}] d\mu \leq \lim_{j\to\infty} \int g_j d\mu - \delta$$

contradicting equality in Fatou's lemma. Since $\{f_n\}$ is now shown to be uniformly integrable and f is integrable it follows that $|f_n - f|$ is uniformly integrable and therefore $\int |f_n - f| d\mu \to 0$.

μ can be σ-finite. Let $\phi > 0$ be integrable. If we define $\lambda(A) = \int \phi(x) d\mu$, then λ is a finite measure and

$$\int f d\mu = \int f \phi^{-1} d\lambda$$

$f_n \phi^{-1} \to f \phi^{-1}$ a.e. and $\int f_n \phi^{-1} d\lambda \to \int f \phi^{-1} d\lambda$. We conclude that $f_n \phi^{-1}$ is uniformly integrable with respect to λ and

$$\int |f_n \phi^{-1} - f \phi^{-1}| d\lambda = \int |f_n - f| \phi^{-1} d\lambda = \int |f_n - f| d\mu \to 0$$

Problem 2i.

$$\begin{aligned} \|x_n - x\|^2 &= < x_n - x, x_n - x > \\ &= < x_n, x_n > - < x_n, x > - < x, x_n > + < x, x > \\ &= 2[1 - < x_n, x >] \to 0 \end{aligned}$$

Problem 2ii. We can assume with out loss of generality that $\int_X |f_n|^2 d\mu = \int_X |f|^2 d\mu = 1$ and use problem 2i. We need to establish that

$$\lim_{n\to\infty} \int f_n(x) g(x) d\mu = \int f(x) g(x) d\mu$$

for all $g \in L_2(\mu)$. We have it for $g = \mathbf{1}_A$. take linear combinations and we have it for simple functions. Simple functions are dense in $L_2(\mu)$. Finally given $g \in L_2(\mu)$, for any $\epsilon > 0$ we can fine a simple function $s(x)$ such that $\|s - g\| = [\int |s(x) - g(x)|^2 d\mu]^{\frac{1}{2}} \leq \epsilon$.

$$\int g(x)[f_n(x) - f(x)] d\mu = \int s(x)[f_n(x) - f(x)] d\mu + \int [g(x) - s(x)][f_n(x) - f(x)] d\mu$$

Therefore

$$\begin{aligned}
\limsup_{n\to\infty} \left| \int g(x)[f_n(x) - f(x)] d\mu \right| &\leq \limsup_{n\to\infty} \int [g(x) - s(x)][f_n(x) - f(x)] d\mu \\
&\leq \limsup_{n\to\infty} [\|g - s\| \cdot \|f_n - f\|] \\
&\leq \limsup_{n\to\infty} [\|g - s\| [\|f_n\| + \|f\|]] \\
&= 2\epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary we are done.

Assignment 6.

Problem 1. Step 1. Start wih a countable collection of disjoint sets $\{A_j\}$ with positive measure. Then functions of the form

$$\sum_j a_j \mathbf{1}_{A_j}(x)$$

provide a $1-1$ correspondence between $\{a_j\} \in \ell_\infty$ and $g(x) = \sum_j a_j \mathbf{1}_{A_j}(x)$ in $L_\infty(\mu)$.

$$\text{esssup } |g(x)| = \sup_j |a_j|$$

Problem 1. Step 2. Consider in ℓ_∞ the subspace

$$E = \left[\xi = \{a_n\} : \Lambda(\xi) = \lim_{n\to\infty} a_n \text{ exists} \right]$$

E is a closed subspace of ℓ_∞ and $\Lambda(\xi)$ is a bounded linear functional on E. By Hahn-banach theorem it can be extended to all of ℓ_∞.

Problem 1. Step 3. Suppose for some $\{p_j\} \in \ell_1$,

$$\Lambda(\xi) = \sum_j a_j p_j$$

Then if we take $\xi_n = \{0, \ldots, 0, 1, 1, \ldots\}$, i.e. n zeros followed by ones, $\Lambda(\xi_n) = 1$ for all n. But for $\{p_j\}$ in ℓ_1 one cannot have

$$1 = \sum_{j=n+1}^{\infty} p_j$$

for all n.

Problem 1. Step 4. We extend the linear functional to $L_\infty(\mu)$ from the subspace of functions of the form

$$\sum_j a_j \mathbf{1}_{A_j}(x)$$

If

$$\Lambda(g) = \int g(x)\phi(x) d\mu$$

for some $\phi \in L_1(\mu)$, then for any

$$g(x) = \sum_j a_j \mathbf{1}_{A_j}(x)$$

$$\begin{aligned} \lambda(g) &= \int [\sum_j a_j \mathbf{1}_{A_j}(x)] d\mu \\ &= \sum_j a_j \mu(A_j) \end{aligned}$$

where $\mu(A_j) = \int_{A_j} \phi(x) d\mu$ and $\{p_j\} = \mu(A_j) \in \ell_1$, providing a contradiction.

Problem 2. Let μ be non-atomic. Then there are sets $\{A_n\}$ that are disjoint and $a_n = \mu(A_n)$ satisfies $0 < a_n < 2^{-n}$ for large n. Consider the function

$$g(x) = \sum_n \mathbf{1}_{A_n} c_n$$

If g is to be in L_{p_1} but not in L_{p_2} with $p_2 > p_1$, we need

$$\sum |c_n|^{p_1} a_n < \infty$$

as well as

$$\sum |c_n|^{p_2} a_n = \infty$$

Take $c_n > 0$ to satisfy

$$c_n^{p_2} = \frac{1}{na_n}$$

Then

$$\sum_n c_n^{p_2} a_n = \sum_n \frac{1}{n} = \infty$$

and

$$\sum_n c_n^{p_1} a_n = \sum_n \frac{1}{n} (na_n)^{1-\frac{p_2}{p_1}} < \infty$$

On the other hand, if $\mu(X)$ is infinite we can find disjoint subsets A_n with $a_n = \mu(A_n) \geq n^2$. Given $p_1 < p_2$ pick

$$c_n^{p_1} = \frac{1}{na_n}$$

so that

$$\sum_n c_n^{p_1} a_n = \sum_n \frac{1}{n} = \infty$$

But now

$$\sum_n c_n^{p_2} a_n = \sum_n \frac{1}{n} (\frac{1}{na_n})^{\frac{p_2}{p_1}-1} < \infty$$

Assignment 5.

Problem 1. Assume

$$\sum_n \int_{A_n} f d\mu < \infty$$

Then for any A with $\mu(A) < \infty$ and $f \geq 0$,

$$\int_A f d\mu = \int_{\cup_n (A \cap A_n)} f d\mu = \sum_n \int_{A \cap A_n} f d\mu \leq \sum_n \int_{A_n} f d\mu$$

so that

$$\sup_{A:\mu(A)<\infty} \int_A f d\mu \leq \sum_n \int_{A_n} f d\mu$$

On the other hand

$$\sum_{1 \leq n \leq N} \int_{A_n} f d\mu = \int_{\cup_{n=1}^N A_n} f d\mu \leq \sup_{A:\mu(A)<\infty} \int_A f d\mu$$

Letting $N \to \infty$,

$$\sum_1^\infty \int_{A_n} f d\mu \leq \sup_{A:\mu(A)<\infty} \int_A f d\mu$$

Hence if either one is finite so is the other and both are equal.

Problem 2. Let $f_n \geq 0$ and $f_n \to f$ a.e. If $\mu(A) < \infty$, then from Fatou's lemma proved for finite measures

$$\int_A f d\mu \leq \liminf_{n\to\infty} \int_A f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$$

Since this is true for every set A wit $\mu(A) < \infty$,

$$\int f d\mu = \sup_{A:\mu(A)<\infty} \int_A f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$$

Assignment 4.

Problem 1i.

$$E = \cup_k \cap_n \{x : f_n(x) \leq k\}$$

Problem 1ii.

$$f^*(x) = \limsup_{n\to\infty} f_n(x) = \inf_{k\geq 1} \sup_{n\geq k} f_n(x)$$

is measurable because

$$\begin{aligned}\{x : f^*(x) \geq a\} &= \cap_{k\geq 1}\{x : \sup_{n\geq k} f_n(x) \geq a\} \\ &= \cap_{k\geq 1} \cap_m \{x : \sup_{n\geq k} f_n(x) > a - \frac{1}{m}\} \\ &= \cap_{k\geq 1} \cap_m \cup_n \{x : f_n(x) > a - \frac{1}{m}\}\end{aligned}$$

Similarly $f_*(x) = \liminf_{n\to\infty} f_n(x) = \sup_{k\geq 1} \inf_{n\geq k} f_n(x)$ is measurable and so is the set

$$\{x : f^*(x) = f_*(x)\}$$

and the restriction of $f^* = f_*$ to this set.

Problem 2. Here μ is a finite measure. If $f_n \to f$ a.e.

$$\mu\Big[\cap_{n\geq 1} \cup_{k\geq n}\{x : |f_k(x) - f(x)| \geq \epsilon\}\Big] = 0$$

By countable additivity since

$$\cup_{k\geq n}\{x : |f_k(x) - f(x)| \geq \epsilon\}$$

is a decreasing sequence of sets

$$\mu[\cup_{k\geq n}\{x : |f_k(x) - f(x)| \geq \epsilon\}] \to 0$$

as $n \to \infty$. But

$$\mu[x : |f_n(x) - f(x)| \geq \epsilon] \leq \mu[\cup_{k\geq n}\{x : |f_k(x) - f(x)| \geq \epsilon\}] \to 0$$

Assignment 3.

Problem 1i. If $b > a$,

$$F(b) - F(a) = \mu[(-\infty, b]] - \mu[(-\infty, a]] = \mu[(a, b]] \geq 0$$

Problem 1ii.

$$\lim_{k\to\infty} F(x + \frac{1}{k}] = \lim_{k\to\infty} \mu[(-\infty, x + \frac{1}{k}]] = \lim_{k\to\infty} \mu[\cap_{k\geq 1}(-\infty, x + \frac{1}{k}]] = \mu[(-\infty, x]] = F(x)$$

Problem 1iii.

$$\lim_{k\to-\infty} F(k) = \lim_{k\to-\infty} \mu[(-\infty, k]] = \lim_{k\to-\infty} \mu[\cap_{k\geq 1}(-\infty, k]] = \mu[\emptyset] = 0$$

$$\lim_{k\to\infty} F(k) = \lim_{k\to\infty} \mu[(-\infty, k]] = \lim_{k\to\infty} \mu[\cup_{k\geq 1}(-\infty, k]] = \mu[R] = 1$$

Problem 2. We define for intervals $(a, b]$ where a can be $-\infty$ and b can be ∞, $\mu[(a, b]] = F(b) - F(a)$. Of course $(a, \infty]$ is the same as (a, ∞). We need to prove that if

$$(a, b] = \cup_j (a_j, b_j]$$

then

$$F(b) - F(a) = \sum_j [F(b_j) - F(a_j)]$$

Since one side is obvious it is inly necessary to prove

$$F(b) - F(a) \leq \sum_j [F(b_j) - F(a_j)]$$

Then by the Caratheodary extension theorem we can extend μ from the semiring of intervals to the Borel σ-field. Because $F(x) \to 0$ as $x \to -\infty$ we can replace a by a finite number a' with $F(a') - F(a) < \epsilon$. Similarly we can replace b if it is ∞ by b' with $F(b) - F(b') < \epsilon$. Using right continuity we can replace $(a_j, b_j]$ by (a_j, b'_j) with $F(b'_j) - F(b_j) \leq \epsilon\, 2^{-j}$. We now have

$$F(b') - F(a') \geq F(b)_F(a) - 2\epsilon$$

and $[a', b'] \subset (a, b]$ is a closed bounded interval. In addition $(a_j, b'_j) \supset (a_j, b_j]$ is an open covering of $[a', b']$ By Heine-Borel theorem there is a finite sub-cover from $\{(a_j, b'_j)\}$ and

$$\sum_j F(b_j) - F(a_j) \geq \sum_j [F(b'_j) - F(a_j)] - \epsilon 2^{-j} \geq [F(b') - F(a')] - \epsilon \geq F(b) - F(a) - 3\epsilon$$

Since $\epsilon > 0$ is arbitrary, countable additivity follows. As for uniqueness μ is determined on the semiring and therefore on the field generated by disjoint union of sets from the semiring, i.e. disjoint union of intervals $(a, b]$. But if two measures agree on a filed they agree on the σ-field generated by the field.