The completion of rationals. Let Q be the set of rational numbers. Consider the set $\mathcal{R}$ of all partitions of Q into two nonempty disjoint sets L and R, with the following properties. Such a partition will be called a Dedekind cut.

1. $Q = L \cup R$

2. $L \cap R = \emptyset$

3. For any $x \in L$ and $y \in R$, $x < y$.

4. There is no $q \in L$ such that $y \leq q$ for all $y \in L$.

We note the following.

1. Any $Q \subset \mathcal{R}$. For any $q \in Q$ we can define

$$L_q = \{x : x < q\} \& R_q = \{y : y \geq q\}$$

Then $[L_q, R_q]$ satisfies all the properties and is a Dedekind cut.

2. Different q lead to different cuts. If $q_1 \neq q_2$ then either $q_1 > q_2$ or $q_2 > q_1$. Assume the former. Then $q_2 \in L_{q_2}$ and $q_2 \notin L_{q_1}$. Conversely if L has a largest element $q \in Q$ then $L = L_q$.

3. If $[L_1, R_1]$ and $[L_2, R_2]$ are two different Dedekind cuts then either $L_1 \subset L_2$ or $L_2 \subset L_1$. Assume $L_1 \not\subset L_2$. The there is q such that $q \in L_1$ but $q \in R_2$. Let $q' \in L_2$. Then $q' < q$ and since $q \in L_1$ so is q'. We say $[L_1, R_2] < [L_2, R_2]$ if $L_1 \subset L_2$. This defines an ordered set $\mathcal{R} = \{[L, R]\}$ that contains Q and extends the order relation in Q.

4. $\mathcal{R}$ is complete. If A is any subset of $\mathcal{R}$ that is bounded above it has a least upper bound. Either a rational q is an upper bound for A or not. Define L as the set of rationals that are not upper bounds and R as those that are. Then $[L, R]$ is seen to be a Dedekind cut. It is not hard to verify that it is in fact the least upper bound for A.