Spectrum of Compact Operators. Let $T : \mathcal{H} \to \mathcal{H}$ be a compact, self adjoint linear operator. Consider the quadratic form $Q(x) =< Tx, x >$. Let

$$\sup_{\|x\|\leq 1} Q(x) = \lambda > 0$$

Then it is attained at some y with $\|y\| = 1$ and $Ty = \lambda y$. There is a sequence x_n with $\|x_n\| \leq 1$ and $< Tx_n, x_n > \to \lambda$. We can assume by taking a subsequence that $x_n \to y$ weakly. Then since T is compact Tx_n will converge to Ty in norm and consequently $Q(x_n) \to Q(y) =< Ty, y >= \lambda$. $\|y\| = c \leq 1$, $\|c^{-1}y\| = 1$ and $Q(c^{-1}y) = c^{-2}\lambda > \lambda$ a contradiction. Therefore $Q(x)$ attains its maximum on $\|x\| = 1$ at y. For any x with $\|x\| = 1$ and $x \perp y$, we have $\|y + tx\| = \sqrt{1 + t^2}$.

$$Q(\frac{y + tx}{\sqrt{1 + t^2}}) \leq Q(y)$$

The derivative with respect to t at 0 is 0.

$$< Tx, y > + < Ty, x >= 0$$

Since T is self adjoint $< Ty, x >= 0$ whenever $< y, x >= 0$. This forces $Ty = cy$ for some c and $< Ty, y >= c = \lambda$. Consider the case

$$\inf_{\|x\|\leq 1} Q(x) = \lambda < 0$$

for negative eigenvalues exhaust all of them on both sides. Any two eigenvectors for different eigenvalues are orthogonal. Let $Tx = \lambda x, Ty = \mu y$ with $\lambda \neq \mu$. Then

$$0 =< Tx, y > - < Ty, x >= (\lambda - \mu) < x, y >$$

Any sequence of orthonormal vectors tends to 0 weakly. $\sum_j | < x, e_j > |^2 \leq \|x\|^2$. $Te_j \to 0$ in norm. If e_j are eigenvectors then $Te_j = \lambda_j e_j$ and $\lambda_j \to 0$.

Example. In $L_2[0, 1]$ with Lebesue measure

$$(Tf)(s) = \int_0^1 \min(s, t) f(t) dt$$

$$g(s) = (Tf)(s) = \int_0^s tf(t)dt + \int_s^1 sf(t)dt$$

$g(0) = 0$ and

$$g'(s) = sf(s) - sf(s) + \int_s^1 f(t)dt = \int_s^1 f(t)dt$$

$g'(1) = 0$ and

$$g''(s) = -f(s)$$

Need to solve $\lambda f''(s) = -f(s)$ with $f(0) = f'(1) = 0$.

$$f(s) = a \cos cs + b \sin cs$$

where $c^2\lambda = 1$. $f(0) = a = 0$. $f'(1) = bc \cos cs = 0$ if $c = (2n + 1)\frac{\pi}{2}$.

Projections. $\mathcal{K} \subset \mathcal{H}$ is a subspace.

$$K^{\perp} = \cap_{x\in\mathcal{K}}\{y :< x, y >= 0\}.$$

$(\mathcal{K}^{\perp})^{\perp} = \mathcal{K}$, $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$, $x = Px + (I - P)x$.

$$\inf_{y\in\mathcal{K}} \|x - y\|^2 = \|(x - Px)\|^2 = \|(I - P)x\|^2$$

$$\mathcal{H} = \oplus_j \mathcal{K}_j$$

$\mathcal{K}_i \perp \mathcal{K}_j$ for $i \neq j$ and if $x \perp \mathcal{K}_|$ for all j then $x = 0$.

Spectral Measures.

Let T be a self-adjoint transformation. Then T is nonnegative definite if $< Tx, x > \geq 0$ for all $x \in \mathcal{H}$.

Theorem. If $p(t)$ is a polynomial with real coefficients and T is self-adjoint then $p(T)$ is well defined and is self-adjoint. More over if $\|T\| = c$ and $p(t) \geq 0$ on $[-c.c]$, then $p(T)$ is non-negative definite.

proof. If T is self adjoint so are all the powers and their linear combinations. If $p(t)$ is a polynomial it can be factored as $[\Pi_i(t - c_i)][\Pi_j[(t - a_j)^2 + b_j^2]$ interms of real and complex roots. All the roots in $[-c.c]$ have to be of even multiplicity and complex roots come in conjugate pairs. The polynomial $p(t)$ can be factored as

$$k[\Pi_{i:c_i\leq -c}(t - c_i)][\Pi_{i:d_i>c}(d_i - t)][\Pi_j[(t - a_j)^2 + b_j^2]$$

or

$$k[\Pi_{i:c_i\geq 0}(t + c + c_i)][\Pi_{i:d_i>0}(c - t + d_i)][\Pi_j[(t - a_j)^2 + b_j^2]$$

with $k > 0$ and $c_i, d_i \geq 0$. It can be rewritten as a linear combination with positive weights of

$$(t + c)^{\alpha}(c - t)^{\beta}(t - a_j)^{2\gamma}, 1$$

We know that $cI - T, cI + T, c^2I - T^2$ and I are positive semi definite. If A is any one of them $< Ay, y >\geq 0$ and y can be $(c - T)^{\alpha}(c + T)^{\beta}(T - a_j)^{\gamma}x$. Makes $p(T)$ positive semidefinite.

Polynomials are dense in $C[-c, c]$. Extend continuously. Non negative linear function. Riesz Representation.

$$< p(T)x, x >= \int_c^c p(t)d\mu_{x,x}(t)$$

μ is a non-negative measure with total mass $\|x\|^2$.

$$\mu_{x,y}(dt) = \frac{1}{2}[\mu_{x+y}(dt) - \mu_{x,x}(dt) - \mu_{y,y}(dt)]$$

$$< p(T)x, y >= \int_c^c p(t) d\mu_{x,y}(t)$$

We can replace $p(t)$ by any bounded measurable function.

$$< f(T)x, y >= \int_c^c f(t) d\mu_{x,y}(t)$$

$$(fg)(T) = f(T)g(T)$$

f can be $\chi_E(t) = \mathbf{1}_E(t)$. Then $\chi_E(T) = \mu(E)$ is a projection onto the eigenspace corresponding to all the eigenvalues in E.

$$T = \int_{-c}^c t d\mu(t)$$

or

$$< Tx, y >= \int_{-c}^c t\mu_{x,y}(dt)$$

Compare it to $T = \sum \lambda_j P_j$. $\mu(dt)$ is a projection valued measure.

Fourier Series.

A complex Hilbert space is a vector space over the field of complex numbers. $a_1x_1 + a_2x_2$ is defined for $a_1, a_2 \in \mathcal{C}$. The inner product $< x, y >$ is linear in x for each y and has the property $< x, y >= \overline{< y, x >}$. $< x, x > \geq 0$ and $\sqrt{< x, x >}$ is a norm under which $\mathcal{H}$ is complete. An orthonormal basis is one such that $< e_i, e_j >= \delta_{i,j}$ for all i, j and the only vector x with $< x, e_j >= 0$ for all j is $x = 0$. Every $x \in \mathcal{H}$ has a representation in terms of an orthonormal basis

$$x = \sum_{j=-\infty}^{\infty} < x, e_j > e_j$$

with $\sum_{j=-\infty}^{\infty} | < x, e_j > |^2 = \|x\|^2$.

The important example is $\mathcal{H} = L_2[0, 2\pi]$, with Lebesgue measure. $\{e_j\}$, $j \in Z$ given by $f_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx}$ where $i = \sqrt{-1}$. Remember that $e^{iax} = \cos ax + i \sin ax$. One can check that

$$< f_j, f_k >= \frac{1}{2\pi} \int_0^{2\pi} f_j(x)\overline{f_k(x)} dx = \delta_{j,k}$$

If you want to stay in the real world you can consider $\frac{1}{\sqrt{2\pi}}$, and $\frac{\sin nx}{\sqrt{\pi}}$, $\frac{\cos nx}{\sqrt{\pi}}$ for $n \geq 1$. It is easy to check that it is an orthonormal set on $\mathcal{H} = L_2[0, 2\pi]$. Why is it complete? Need to prove linear combinations are dense in $\mathcal{H}$. We can identify 0 and 2π and view them as periodic functions on $[0, 2\pi]$ or functions on the circle S. It is therefore enough to prove that the linear combinations are dense in the uniform topology in the space of continuous functions on S. Linear combinations is an algebra (trig identities). Contains constants. Distinguishes points. Apply Stone-Weierstrass.

Convergence Properties of Fourier Series.

$f(x)$ is a complex valued function on $[0,2\pi]$ with $\int_0^{2\pi}|f(x)|^2dx<\infty$

$$c_n=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}f(x)e^{-inx}dx$$

One expects

$$f(x)=\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}c_ne^{inx}$$

1. We saw that $\{e^{inx}\}$ is a complete orthonormal set. Therefore

$$\int_0^{2\pi}|\sum_{-k}^{\ell}c_ne^{inx}-f(x)|^2dx\to 0$$

as $k,\ell\to\infty$. If f is smooth on S, i.e with f and some derivatives matching at 0 and 2π we can integrate by parts and

$$c_n=\frac{(-i)^d}{n^d}\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}f^{\{d\}}(x)e^{-inx}dx$$

One expects rapid convergence for smooth functions.

2. Let us compute

$$\begin{aligned}
s_N(x)&=\frac{1}{\sqrt{2\pi}}\sum_{n=-N}^{N}c_ne^{inx}\\
&=\frac{1}{2\pi}\sum_{n=-N}^{N}\int_0^{2\pi}e^{-iny}f(y)e^{inx}dy\\
&=\frac{1}{2\pi}\int_0^{2\pi}f(y)\sum_{n=-N}^{N}e^{in(x-y)}dy\\
&=\frac{1}{2\pi}\int_0^{2\pi}f(y)\sum_{n=-N}^{N}e^{-iN(x-y)}\frac{[e^{i(2N+1)(x-y)}-1]}{e^{i(x-y)}-1}dy\\
&=\frac{1}{2\pi}\int_0^{2\pi}f(y)\frac{\sin(N+\frac{1}{2})(x-y)}{\sin\frac{x-y}{2}}dy\\
&=\frac{1}{2\pi}\int_0^{2\pi}f(x-z)\frac{\sin(N+\frac{1}{2})z}{\sin\frac{z}{2}}dz
\end{aligned}$$

$$s_N(x)-f(x)=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(x-z)-f(x)}{\sin\frac{z}{2}}(\sin(N+\frac{1}{2})z)dz$$

If $\frac{f(x-z)-f(x)}{z}$ is integrable near 0, then by Riemann-Lebesgue lemma the integral goes to zero. $|f(y)-f(x)| \leq C|x-y|^\alpha$ is enough.

Riemann Lebesgue Lemma.

If $f(x)$ is integrable $|\int_{-\infty}^{\infty} f(x)e^{itx}dx| \to 0$ as $t \to \infty$. If f is smooth and has compact support integration by parts will do it. f can be approximated in L_1 by smooth function g such that $\int |f-g|dx < \epsilon$. Then $|\int fe^{itx}dx - \int ge^{itx}dx| \leq \int |f-g|dx < \epsilon$.

Fejer Kernel.

$$\frac{s_0 + s_1 + \cdots + s_{N-1}}{N} = \int f(y)k_N(x-y)dy$$

where

$$\begin{aligned} k_N(x) &= \frac{1}{2\pi N \sin\frac{x}{2}}[\sin\frac{x}{2} + \cdots + \sin(N-\frac{1}{2})x] \\ &= \frac{1}{2\pi N}\frac{\sin^2\frac{Nx}{2}}{\sin^2\frac{x}{2}} \end{aligned}$$

$k_N(x) \geq 0, \int_0^{2\pi} k_N(x)dx = 1$, and $k_N(x) \to 0$ if $x \neq 0$. Approximate identity

$$\int k_N(x-y)f(y)dy \to f(x)$$

uniformly for continuous f. In L_p for $f \in L_p$.

Lemma.

$$\|\int k(y-x)f(x)dx\|_p \leq \|f\|_p$$

if $k \geq 0, \int kdx = 1$ on S or R.