15 Solutions

- $$\frac{\partial \log p}{\partial \theta} = \frac{-2(x-\theta)}{1+(x-\theta)^2}$$

  $$I(\theta) = \frac{1}{\pi}\int \left[\frac{-2(x-\theta)}{1+(x-\theta)^2}\right]^2 \frac{1}{1+(x-\theta)^2}dx = \frac{4}{\pi}\int \frac{x^2}{(1+x^2)^3}dx = \frac{1}{2}$$

- $$f(\theta, x_1, \ldots, x_n) = \theta^n \exp[-\theta \sum x_i\ ]$$

  $$\frac{\partial \log f(\theta, x_1, \ldots, x_n)}{\partial \theta} = \frac{n}{\theta} - \sum_i x_i$$

  $$\hat{\theta} = \frac{n}{\sum x_i}$$

  $$E[\hat{\theta}] = \int_0^\infty \frac{n}{t}\frac{\theta^n}{\Gamma(n)}\exp[-\theta\, t]t^{n-1}dt = \frac{n\,\theta}{n-1}$$

  It is biased and $\hat{t} = \frac{n-1}{n}\hat{\theta}$ is unbiased.

  $$Var(\hat{t}) = (n-1)^2\theta^2\left[\frac{1}{(n-1)(n-2)} - \frac{1}{(n-1)^2}\right] = \frac{\theta^2}{n-2} \geq \frac{\theta^2}{n} = \frac{1}{nI(\theta)}$$

  It is asymptotically efficient.

- MLE is the Median. By symmetry it is unbiased.

  $$\frac{\partial \log f}{\partial \theta} = -\text{sign}(x-\theta)$$

  $$I(\theta) = 1$$

  Cramer-Rao lower bound is $\frac{1}{n}$. Asymptotic variance of the MLE is

  $$\frac{4}{n[f(\theta)]^2} = \frac{1}{n}$$

- A statistic is $T(0), \ldots, T(n)$. Unbiased means

$$\sum_j \binom{n}{j} \left[\frac{3}{4}\right]^j \left[\frac{1}{4}\right]^{n-j} T(j) = \frac{3}{4}$$

and

$$\sum_j \binom{n}{j} \left[\frac{1}{4}\right]^j \left[\frac{3}{4}\right]^{n-j} T(j) = \frac{1}{4}$$

Two equations. $n+1$ unknowns. Lots of solutions. Easy to construct unbiased estimators with variance that is very small. For example if n is odd $T(x) = a$ if $x < \frac{n}{2}$ and $T(x) = b$ if $x > \frac{n}{2}$ can be made unbiased by proper choice of a and b. If we denote by

$$p_n = \sum_{j < \frac{n}{2}} \binom{n}{j} \left[\frac{3}{4}\right]^j \left[\frac{1}{4}\right]^{n-j} = \sum_{j > \frac{n}{2}} \binom{n}{j} \left[\frac{1}{4}\right]^j \left[\frac{3}{4}\right]^{n-j}$$

we need

$$ap_n + b(1-p_n) = \frac{1}{4}$$

and

$$a(1-p_n) + bp_n = \frac{3}{4}$$

giving us $a = \frac{4p_n - 3}{8p_n - 4}$ and $b = \frac{4p_n - 1}{8p_n - 4}$. The variance is given by

$$\sigma^2 = p_n(a - \frac{1}{4})^2 + (1-p_n)(b - \frac{1}{4})^2$$

and is seen to be very very small for large n.

- The log-likelihood is

$$-\frac{n}{2}\log\theta - \frac{1}{2\theta}\sum(x_i - \theta)^2 = -\frac{n}{2}\log\theta - \frac{1}{2\theta}\sum x_i^2 + \sum_i x_i - \frac{n}{2}\theta$$

Clearly $U_n = \frac{1}{n}\sum x_i^2$ is sufficient and the likelihood equation is

$$-\frac{1}{\theta} + \frac{U_n}{\theta^2} - 1 = 0$$

or

$$\theta^2 + \theta = U_n$$

This gives

$$\theta_n = -\frac{1}{2} + \sqrt{\frac{1}{4} + U_n}$$

which is consistent because

$$-\frac{1}{2} + \sqrt{\frac{1}{4} + \theta^2 + \theta} = \theta$$

Has an asymptotic variance

$$\frac{1}{n} \text{ var } (x^2)[f'(\theta^2 + \theta)]^2$$

with

$$f(y) = -\frac{1}{2} + \sqrt{\frac{1}{4} + y}$$

and

$$f'(\theta^2 + \theta) = \frac{1}{2\theta + 1}$$

On simplification this reduces to $\frac{1}{n}\frac{2\theta^2}{(1+2\theta)^2}$. The Cramer-Rao lower bound is eaxctly the same. The eficiency of the mean with variance $\frac{\theta}{n}$ is given by $\frac{2\theta}{(1+2\theta)^2}$.

- The log-likelihood function is

$$-n \log \Gamma(p) - \sum x_i + (p-1) \sum \log x_i$$

Tha likelihood equation is

$$\frac{\Gamma'(p)}{\Gamma(p)} = G(p) = \frac{1}{n} \sum \log x_i$$

and the MLE is

$$\hat{\theta}_n = G^{-1}(\frac{1}{n} \sum \log x_i)$$

It is consistent because

$$\hat{\theta}_n \to G^{-1}(m)$$

where

$$m = \int \frac{1}{\Gamma(p)} e^{-x} x^{p-1} \log x dx = G(p)$$

and

$$G^{-1}(G(p)) = p$$

$\hat{\theta}_n$ is asymptotically normal with variance $\frac{\text{var } (\log x_i)}{n[G'(p)]^2}$. The quantity $I(p)$ is calculate easily as

$$I(p) = E[(\log x - G(p))^2] = \text{ var } (\log x) = G'(p)$$

- To test $f_0(x) = 2x$ against $f_1(x) = 2(1-x)$ the critical region is $\frac{1-x}{x} > c$ or $x < c$. Size is $\int_0^c 2xdx = c^2 = \alpha$ or $c = \sqrt{\alpha}$. Power is calculated to be

$$\int_0^{\sqrt{\alpha}} 2(1-x)dx = 2\sqrt{\alpha} - \alpha$$

- The critical region can be any subset of $[0, \frac{1}{2}]$ if $\alpha < \frac{1}{2}$ or the entire $[0, \frac{1}{2}]$ along with a subset of $[\frac{1}{2}, 1]$ if $\alpha > \frac{1}{2}$. The power is 2α if $\alpha < \frac{1}{2}$ and 1 if $\alpha > \frac{1}{2}$.

- The critical region is of the form $\sqrt{n}|\bar{x}_n| > c$ and $c = 1.96$ from the tables. Power at $\mu = 1$ is $P[|z - \sqrt{n}| > 1.96]$ is essentially $P[z < \sqrt{n} - 1.96]$ and this is .95 if $\sqrt{n} > 1.96 + 1.64 = 3.61$ or $n > (3.61)^2 = 14$