New York University
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G63.2020/G22.2421: Numerical Methods II — *Handout #2*

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Problem Set 2

Problem 1

This problem requires the application of Newton's iteration in a non-ODE setting. Let $\{x_j, j = 1, 2, \ldots, n\}$ be the n real roots in $(-1, 1)$ of the Legendre polynomial $p_n(x)$ of degree n. It is well-known that the Chebyshev node $t_j = \cos[(n-j+1/2)\cdot\pi/n]$, which are the n roots of the Chebyshev polynomial $T_n(x)$ of degree n, is fairly close to the Legendre roots x_j.
(a) Apply Newton's iteration to find x_j to double precision with t_j as the starting value (use the recursion to evaluate $p_n(x)$: $p_0(x) = 1$, $p_0(x) = x$, and $p_{k+1}(x) = [(2k+1)\cdot x\cdot p_k(x) - k\cdot p_{k-1}(x)]/(k+1)$; derive a similar recursion to evaluate $p'_k(x)$; compute only half of the roots $x_j < 0$ (or > 0) and obtain the other half by symmetry since p_n is either odd if n is odd, or is even if n is even).
(b) Verify quadratic convergence of the Newton's iteration.
(c) For $n = 2$ and 8, print out the roots.

Remark: There is an exact formula (see Analysis of Numerical Methods, Isaacson, page 334) to calculate the Gaussian quadrature weights

$$w_j = \frac{2}{(1-x_j^2)\cdot[p'_n(x_j)]^2}. \tag{1}$$

Table 1 shows the nodes and weights for $n = 10$.

Table 1: Weights and Nodes for Gaussian Quadrature, $n = 10$

i	Weights w_i	Nodes x_i	Chebyshev Nodes t_i
1	0.66671344308688E-01	-0.97390652851717E+00	-0.98768834059514E+00
2	0.14945134915058E+00	-0.86506336668898E+00	-0.89100652418837E+00
3	0.21908636251598E+00	-0.67940956829902E+00	-0.70710678118655E+00
4	0.26926671931000E+00	-0.43339539412925E+00	-0.45399049973955E+00
5	0.29552422471475E+00	-0.14887433898163E+00	-0.15643446504023E+00
6	0.29552422471475E+00	0.14887433898163E+00	0.15643446504023E+00
7	0.26926671931000E+00	0.43339539412925E+00	0.45399049973955E+00
8	0.21908636251598E+00	0.67940956829902E+00	0.70710678118655E+00
9	0.14945134915058E+00	0.86506336668898E+00	0.89100652418837E+00
10	0.66671344308688E-01	0.97390652851717E+00	0.98768834059514E+00

Problem 2

For the choice of parameters $k = 30, \mu = 0.80$, solve the initial value problem

$$\begin{cases} y' = i\cdot k\{y^2 - [1+\mu\cdot\sin(5t)]\}, & 0 \le t \le 2\pi, \\ y(0) = 1, \end{cases} \tag{2}$$

by implementing the following two schemes on a equispaced grid with $h = 2\pi/N$, $t_n = n\cdot h$. Check their rates of convergence numerically, and plot the numerical solutions as function of $t \in [0, 2\pi]$.

(a) The fourth order (four-stage) ERK method given on page 41;
(b) the fourth order (two-stage) IRK method given on page 47.

Remark: Use Newton's method (6.7) on page 95 to solve nonlinear equations for the IRK method (note that there is a typo there, h is missing; h should multiply $\partial g/\partial w$). You are required to use the second order scheme

$$y_{n+1} = y_n + i \cdot hk\{y_n \cdot y_{n+1} - [1 + \mu \cdot \sin(5t_{n+1/2})]\} \tag{3}$$

as the predictor to produce an initial guess for the Newton iteration, and stop the iteration when the relative error is less than $\epsilon = 10^{-8}$.

Problem 3

Using the Taylor expansion or something similar, prove that the approximation error of (3) to (2) is second order in h. You may denote $[1 + \mu \cdot \sin(5t)]$ by g(t), and assume that g is analytic in t. Hint: show that the local error is $O(h^3)$.

Problem 4

Exercise 1.5 on page 17. Hint: use Taylor expansion to check local errors.