Abstract. The analytical treatment of one-dimensional inverse scattering problems for the Helmholtz equation mainly employs either the Riccati equation for the reflection coefficients (see [8], [9], [7]), or Marchenko integral equations (see [3], [4], [2]). The Riccati equation formulation supports such inversion schemes as the trace formula method [6], [7]), [5], and other layer stripping method [8]. Among the main advantages in the Riccati approach for the numerical inversion of the Helmholtz equation are their efficiency and stability. When there are more than one parameters to be recovered (such as the permittivity, conductivity, and permeability), a single reflection coefficient will not be sufficient for the inversion, and the Riccati equation must be extended to the scattering matrix in order to generalize the trace formula. We develop, derive and summarize some of the fundamental operations associated with the scattering matrices for the solution of forward and inverse scattering problems for the one-dimensional Helmholtz equation.

1 Introduction

The subject of the paper is the scattering problem for the Helmholtz equation

$$\phi''(x,k) + k^2(1+\tilde{q}(x))\phi(x,k) = 0. \tag{1}$$

where $\tilde{q} \in C^m(\mathbb{R}^1)$ is complex-valued function of the form $\tilde{q}(x) = q(x) + ip(x,k)$ for real q and p. In order to reconstruct the two parameter q and p, more than one entry of the scattering matrix (see Section 3) must be employed. As is well-known, the scattering matrix is a classical object used in the analysis of the forward and inverse scattering problems. We will derive a Riccati equation for the scattering matrix, develop systematically and summarize several fundamental properties of the scattering matrices, to be used for the reconstruction of q and p. The utility of these operations and properties is not restricted to this particular inverse problem; they will be useful in the inverse problems to recover the density ρ and the index of refraction n of the Helmholtz equation

$$\rho(x)\left[\frac{1}{\rho(x)}\phi'(x,k)\right]' + k^2 n^2(x)\phi(x,k) = 0, \tag{2}$$

as well as in fast solvers for two-point boundary value problems for a general second order ODE. The paper is organized as follows. In Sections 2 and 3, we introduce the scattering problem and define the scattering matrices. The merging and splitting operations are specified in Section 4. Sections 5 and 6 are devoted to the construction of scattering matrices and derivation of their governing Riccati equations. Formulae for the recovery of a scattering matrix of a subinterval is provided in Section 7; they are essential for the inversion of equation (1). Finally, in Section 8, three more basic operations are presented for the analysis and compuation of the scattering problems, including the proof of a result by Jaulent and Jean [2] which relates the scattering matrices of dissipative and active media.

2 The scattering problems

We assume that $\tilde{q}$ is supported in a finite interval $[a, b]$. We will consider the scattering solutions of (1) in the form

$$\phi(x,k) = \phi_0(x,k) + \psi(x,k), \quad \phi_0(x,k) = e^{\pm ikx}. \tag{3}$$

where ϕ_0 is referred to as the incident field, ψ the scattered field, which is required to be subject to the outgoing radiation condition

$$\psi'(a,k) + ik\psi(a,k) = 0, \quad \psi'(b,k) - ik\psi(b,k) = 0. \tag{4}$$

The scattered field satisfies the equation

$$\psi''(x,k) + k^2(1+\tilde{q}(x))\psi(x,k) = -k^2\tilde{q}(x)\phi_0(x,k) \tag{5}$$

Equivalent, (4) and (5) can be reformulated as the Lippmann-Schwinger equation

$$\psi(x) = -k^2 \int_\Omega G_k(x,\xi)\tilde{q}(\xi)(\phi_0(\xi) + \psi(\xi))d\xi, \tag{6}$$

with G_k the free-space Green's function defined by the formula

$$G_k(x,y) = \frac{1}{2ik}\begin{cases} e^{ik(y-x)}, & x \le y, \\ e^{ik(x-y)}, & x \ge y. \end{cases} \tag{7}$$

In the case of

$$\tilde{q}(x) = q(x) + \frac{i}{k}m(x), \tag{8}$$

the Helmholtz equation (1) can be written

$$\phi''(x,k) + k^2(1+q(x))\phi(x,k) + ikm(x)\phi(x,k) = 0, \tag{9}$$

where we assume that $q \ge q_0 > -1$ and $m \ge 0$. It is well-known that the forward scattering problem is well-posed for $q, m \in C^1$, and that analyticity and high-frequency asymptotics of the reflection coefficients (see Remark 3.2 for their technical definition) have been well-established [1]. The inverse scattering problem is also well-posed [2]; the two reflection coefficients and one transmission coefficient

$$\{\ r_+(k),\ r_-(k),\ t_-(k),\ k \in \mathbb{R}^1\ \} \tag{10}$$

determine the two functions m and q uniquely. If $m(x)$ has mixed sign or is negative, however, bound states may develop [1]. In the presence of the bound states, (10) cannot uniquely determine the two functions. We will refer to the case $m \ge 0$ as dissipative, and to the case $m \le 0$ as active.

In the context of acoustic scattering in layered medium, $n = \sqrt{1+\tilde{q}}$, whose real part is required to positive, is the index of refraction; $m \ge 0$ characterizes dissipation of the medium, and (9) models wave scattering in a lossy material. In the context of electromagnetic scattering, n is the permittivity, $m(x)$ is the conductivity which is responsible for the dissipation of electromagnetic energy.

3 Scattering matrices

In the classical definition of the scattering matrix, the two reflection coefficients are defined via the scattered fields whereas the two transmission coefficients are defined via the total fields. Consequently, the 2-by-2 matrix is unitary if $\tilde{q}$ is real-valued. We will define a slightly different scattering matrix whose four entries are all based on the scattered fields; the entries are also properly scaled for their causality. We found the new definition easier and more natural to sustains the two basic operations on the scattering matrices (i) Merging of two scattering matrices (ii) Riccati equation for a scattering matrix. In the presence of p or m for which $\tilde{q}$ is complex-valued, the unitarity of the classical scattering matrix is no longer valid, and will be replaced by an identity which links scattering matrix of the dissipative case $m \geq 0$ with that of the active case $m \leq 0$. These properties of the scattering matrix will play an crucial role in the analysis and solution of the inverse problem where more than one parameters are to be recovered.

3.1 Reflection and transmission coefficients

From now on we assume that the scatterer is supported in the finite interval $[0,1]$. Consider the scattering problem restricted to a subinterval $[a,b]$ of $[0,1]$ by setting m and q zero outside $[a,b]$. There are four parameters that completely describe a scattering event. They are the two reflection coefficients and two transmission coefficients which we define now.

To the subinterval $[a,b]$, a right-going incident field of unit strength is chosen as $e^{ik(x-a)}$ which induces a scattered field ψ. Outside $[a,b]$,

$$\psi(x) = \begin{cases} r^{+}_{[a,b]}(k)e^{-ik(x-a)} & \text{if } x \leq a, \\ t^{+}_{[a,b]}(k)e^{ik(x-b)} & \text{if } x \geq b. \end{cases} \tag{11}$$

Likewise, to the subinterval $[a,b]$, a left-going incident field of unit strength is chosen as $e^{-ik(x-b)}$ which induces a scattered field ψ. Outside $[a,b]$,

$$\psi(x) = \begin{cases} t^{-}_{[a,b]}(k)e^{-ik(x-a)} & \text{if } x \leq a, \\ r^{-}_{[a,b]}(k)e^{ik(x-b)} & \text{if } x \geq b. \end{cases} \tag{12}$$

Naturally, $r^{\pm}_{[a,b]}(k)$ are referred to as the reflection coefficients, and not so naturally, $t^{\pm}_{[a,b]}(k)$ are referred to as the transmission coefficients; see the following remark.

Remark 3.1 *While the definition of the reflection coefficients conforms with existing literature, our definition of the transmission coefficients is different from the standard; here the transmission coefficients specify the scattered fields whereas the standard specify the total fields.*

Remark 3.2 *Our scattering data $r_{\pm}(k)$ $t_{-}(k)$ (see (10)) are defined as the two reflection coefficients $r^{\pm}_{[a,b]}(k)$ and the transmission coefficient $t^{-}_{[a,b]}(k)$ when $[a,b]$ becomes*

$[0,1]$, *namely,*

$$r_\pm(k) = r^\pm_{[0,1]}(k), \;\; t_-(k) = t^-_{[0,1]}(k). \tag{13}$$

3.2 Scattering matrices

We will organize the four coefficients $r^\pm_{[a,b]}(k)$ and $t^\pm_{[a,b]}(k)$ by arranging them as a 2-by-2 matrix, which will be referred to as the scattering matrix. This scattering matrix differ from the classical; it is not unitary even if $m = 0$. It is convenient, however, to use this scattering matrix to merge scatterers in non-overlapping sub-intervals, as is required for deriving the ODEs (see Remark 3.2). By our definition, the incident field to $[a,b]$ has the form

$$\phi_0(x,k) = \alpha^+_{[a,b]}(k)e^{ik(x-a)} + \alpha^-_{[a,b]}(k)e^{-ik(x-b)}, \tag{14}$$

namely, $\alpha^+_{[a,b]}(k)$ is the coefficient of the right-going field and $\alpha^-_{[a,b]}(k)$ is the coefficient of the left-going field. Since we are looking at the scattering behavior of the chunk $[a,b]$ alone, it is assumed that outside the chunk the medium is homogeneous: $q = m = 0$; therefore, the scattered field outside assume the form

$$\psi(x) = \begin{cases} \beta^-_{[a,b]}(k)e^{-ik(x-a)} & \text{if } x \le a, \\ \beta^+_{[a,b]}(k)e^{ik(x-b)} & \text{if } x \ge b. \end{cases} \tag{15}$$

Introducing the coefficients of the incident and scattered fields

$$\text{coeff}_{[a,b]}\{\phi_0\} = \begin{bmatrix} \alpha^+_{[a,b]}(k) \\ \alpha^-_{[a,b]}(k) \end{bmatrix}, \tag{16}$$

$$\text{coeff}_{[a,b]}\{\psi\} = \begin{bmatrix} \beta^-_{[a,b]}(k) \\ \beta^+_{[a,b]}(k) \end{bmatrix}, \tag{17}$$

we define the scattering matrix $S_{[a,b]}(k) \in \mathbb{C}^{2\times 2} : \text{coeff}_{[a,b]}\{\phi_0\} \mapsto \text{coeff}_{[a,b]}\{\psi\}$ by the formula

$$\begin{bmatrix} \beta^-_{[a,b]}(k) \\ \beta^+_{[a,b]}(k) \end{bmatrix} = S_{[a,b]}(k) \begin{bmatrix} \alpha^+_{[a,b]}(k) \\ \alpha^-_{[a,b]}(k) \end{bmatrix}. \tag{18}$$

Evidently, the four coefficients $r^\pm_{[a,b]}(k)$ and $t^\pm_{[a,b]}(k)$ appear as entries of the scattering matrix

$$S_{[a,b]}(k) = \begin{bmatrix} r^+_{[a,b]}(k) & t^-_{[a,b]}(k) \\ t^+_{[a,b]}(k) & r^-_{[a,b]}(k) \end{bmatrix}. \tag{19}$$

Lemma 3.3 *The scattering matrix is translation invariant: if the support interval $[a,b]$ of the scatterer is shifted to $[a+d, b+d]$, the scattering matrix S remains the same. Furthermore, if the support interval is flipped, the scattering matrix is flipped twice: one on the rows and the other on the column; in other words, the scattering matrix S^f of the flipped scatterer is given by*

$$S^f_{11} = S_{22}, \quad S^f_{22} = S_{11}, \quad S^f_{12} = S_{21}, \quad S^f_{21} = S_{12}. \tag{20}$$

3.3 Restriction, extension, and translation operators

In this section we develop three basic operations on the scattering coefficients. These operators will be required in merging and splitting procedures which are useful not only in deriving the Riccati equations for the scattering matrices but also in the design of fast and high-order solvers for the general two-point boundary value problems reformulated as second kind integral equations.

The three linear operators which we now introduce, each being a 2-by-2 matrix, are related to the following three situations.

(a) An incident field to an interval is also an incident field to its sub-interval (Restriction).

(b) A scattered field generated from an interval can be regarded as a scattered field generated from an interval containing the former (Extension).

(c) A scattered field generated from an interval can also be regarded as an incident field outside the interval (Translation).

Remark 3.4 *In the sequel, a scattered field in an interval outside where it is generated will always be regarded as an incident field.*

Definition 3.5 *Suppose $a \leq c < d \leq b$ so that $[c, d]$ is contained in $[a, b]$. The four 2-by-2 matrices $R,\ E,\ J_1,\ J_2$ defined by the formulae*

$$R = E = \begin{bmatrix} e^{ik(c-a)} & 0 \\ 0 & e^{ik(b-d)} \end{bmatrix}, \quad J_1 = J_2^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{21}$$

will be referred to as the extension, restriction, left translation, and right translation matrices, respectively.

The next lemma follows immediately from the definition.

Lemma 3.6 *Under the assumption of Definition 3.5, suppose that ϕ_0 is an incident field to $[a, b]$, and ψ is a scattered field from $[c, d]$. Then*

$$\mathit{coeff}_{[c,d]}\{\phi_0\} = R \cdot \mathit{coeff}_{[a,b]}\{\phi_0\}, \quad \mathit{coeff}_{[a,b]}\{\psi\} = E \cdot \mathit{coeff}_{[c,d]}\{\psi\}, \tag{22}$$

$$\mathit{coeff}_{[a,c]}\{\psi\} = J_1 \cdot \mathit{coeff}_{[c,d]}\{\psi\}, \quad \mathit{coeff}_{[d,b]}\{\psi\} = J_2 \cdot \mathit{coeff}_{[c,d]}\{\psi\}, \tag{23}$$

where in (23) the scattered field ψ outside $[c, d]$–its interval of generation–is regarded as an incident field; therefore, for example, $\mathit{coeff}_{[a,c]}\{\psi\}$ is understood as the coefficients of ψ as an incident field to the interval $[a, c]$.

4 Merging two concatenate intervals

We now develop the merging and splitting formulae for the scattering problem, and restrict our discussion to the special case of two concatenate intervals, although the merging and splitting procedures are equally well-defined on two disjoint intervals.

The interval $[a,b]$ is divided into two sub-intervals by a point $c \in (a,b)$, and for given scattering matrices for $I_1 = [a,c]$ and $I_2 = [c,b]$, we will merge them to obtain the scattering matrix for $[a,b]$. Assume that the scatterer in $[a,b]$ is excited by an incident field of the form (14). Denote by ψ_{ij} the restriction on I_i of the scattered field ψ_j generated from I_j, namely,

$$\psi_{ij}(x) = \psi_j(x), \quad x \in I_i, \quad 1 \le i \ne j \le 2 \tag{24}$$

Remark 4.1 *This distinction is made here to highlight that ψ_{ij} will be regarded as an incident field to I_i whereas $\psi_j(x)$ still denotes the scattered field from I_j, see Remark 3.4.*

Then the fundamental law of multiple scattering states that (i) the scattered field ψ from $[a,b]$ is the superposition of the two scattered fields ψ_j

$$\psi(x) = \psi_1(x) + \psi_2(x), \quad x \in \mathbb{R}^1 \tag{25}$$

and that (ii) the total incident field ϕ_j to a scatterer is the superposition of the original incident field ϕ_0 and the scattered field from the other scattering bodies

$$\phi_j(x) = \phi_0(x) + \psi_{ji}(x), \quad x \in I_j, \quad 1 \le i \ne j \le 2 \tag{26}$$

To simplify our discussion, we introduce the following notation.

$$S = S_{[a,b]}(k), \quad S_1 = S_{[a,c]}(k), \quad S_2 = S_{[c,b]}(k), \tag{27}$$

$$\Phi_0 = \mathrm{coeff}_{[a,b]}\{\phi_0\}, \quad \Psi = \mathrm{coeff}_{[a,b]}\{\psi\}, \tag{28}$$

$$\Phi_1 = \mathrm{coeff}_{[a,c]}\{\phi_1\}, \quad \Psi_1 = \mathrm{coeff}_{[a,c]}\{\psi_1\}, \tag{29}$$

$$\Phi_2 = \mathrm{coeff}_{[c,b]}\{\phi_2\}, \quad \Psi_2 = \mathrm{coeff}_{[c,b]}\{\psi_2\}, \tag{30}$$

$$R_1 = E_1 = \begin{bmatrix} 1 & 0 \\ 0 & e^{ik(b-c)} \end{bmatrix}, \quad R_2 = E_2 = \begin{bmatrix} e^{ik(c-a)} & 0 \\ 0 & 1 \end{bmatrix}, \tag{31}$$

From the definition of scattering matrix,

$$\Psi = S \cdot \Phi_0, \quad \Psi_1 = S_1 \cdot \Phi_1, \quad \Psi_2 = S_2 \cdot \Phi_2. \tag{32}$$

The next lemma reformulates the fundamental law of multiple scattering (see (25), (26)); it follows immediately from Lemma 3.6 and (32).

Lemma 4.2 *In the form of coefficients, the fundamental law of multiple scattering states that, for the scattered fields,*

$$\Psi = E_1\Psi_1 + E_2\Psi_2; \tag{33}$$

and that, for the incident fields,

$$\Phi_1 = R_1\Phi_0 + J_1S_2\Phi_2, \tag{34}$$

$$\Phi_2 = R_2\Phi_0 + J_2S_1\Phi_1. \tag{35}$$

Definition 4.3 *The 4-by-2 matrix defined by the formula*

$$S_p = \begin{bmatrix} I & -J_1 S_2 \\ -J_2 S_1 & I \end{bmatrix}^{-1} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \tag{36}$$

will be referred to as the splitting matrix, provided that the inverse exists.

The solution of the 2-by-2 linear system (34) and (35) for $\Phi_1, \ \Phi_2$ establishes the next theorem.

Theorem 4.4 *(***Splitting the incident field***) Suppose that Φ_0 is the coefficients of the incident field ϕ_0 upon $[a, b]$, and that $\Phi_j, \ j = 1, 2$ are the coefficients of the total incident fields $\phi_j, \ j = 1, 2$ upon the two sub-intervals $[a, c], \ [c, b]$, respectively. Then*

$$\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = S_p \cdot \Phi_0. \tag{37}$$

The merging operation of scattering matrices follows directly from (33).

Theorem 4.5 *(***Merge the scattering matrices***) Given the scattering matrices S_1 and S_2 for the scattering bodies $[a, c]$ and $[c, b]$, the scattering matrix S for $[a, b]$ can be obtained via the formula*

$$S = [E_1 S_1 \ \ E_2 S_2] \cdot S_p. \tag{38}$$

5 Scattering matrix of a narrow interval

Our preceding discussion bears no concern on any properties of the scatterer; the only assumption made there is that the underlying equation in the free space is the Helmholtz equation of constant coefficient: $\phi'' + k^2 \phi = 0$. Now in this section and before we derive the Riccati equations for the scattering matrices in the next section, we will call forth and deal directly with the scatterer and use it to build the scattering matrix. Note that (9) can obviously be regarded as a Helmholtz equation with an effective scatterer

$$\tilde{q}(x) = q(x) + \frac{i}{k} m(x). \tag{39}$$

When the interval $[x_0, x_0 + h]$ is illuminated by an incident field ϕ_0, the scattered field ψ, which satisfies the Lippmann-Schwinger equation

$$\psi(x) + k^2 \int_{x_0}^{x_0+h} G_k(x, \xi) \tilde{q}(\xi) \psi(\xi) d\xi = -k^2 \int_{x_0}^{x_0+h} G_k(x, \xi) \tilde{q}(\xi) \phi_0(\xi) d\xi, \tag{40}$$

can be calculated to the second order of h by the Born approximation

$$\psi(x) = -k^2 \int_{x_0}^{x_0+h} G_k(x, \xi) \tilde{q}(\xi) \phi_0(\xi) d\xi + O(h^2) = \frac{ik}{2} \int_{x_0}^{x_0+h} e^{ik|x-\xi|} \tilde{q}(\xi) \phi_0(\xi) d\xi + O(h^2)$$

The incident field are in the form $\phi_0(x) = e^{ik(x-x_0)}$ or $\phi_0(x) = e^{-ik(x-x_0-h)}$; therefore $\phi_0 = 1 + O(h)$, and

$$\begin{aligned} \psi(x) &= O(h^2) + \frac{ik}{2}\int_{x_0}^{x_0+h} e^{ik|x-\xi|}\tilde{q}(\xi)d\xi \\ &= O(h^2) + \frac{ikh}{2}\tilde{q}(x_0)\left\{ \begin{array}{ll} e^{-ik(x-x_0)}, & x \leq x_0 \\ e^{ik(x-x_0-h)}, & x \geq x_0 + h \end{array} \right. \end{aligned}$$

from which follows immediately

Lemma 5.1 *To the second order of h, the scattering matrix $S_{[x_0,x_0+h]}(k)$ for the narrow interval $[x_0, x_0 + h]$ is*

$$S_{[x_0,x_0+h]}(k) = \frac{ikh}{2}\tilde{q}(x_0)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + O(h^2) \tag{41}$$

6 Riccati equations

In this section we assume that $[a, b]$ is the support of the scatterer, although our discussion is valid for arbitrary a, b. Let $x \in (0, 1)$ so that it plays the role of c in the preceding sections. Evidently, there are eight functions of x in the two scattering matrices of the two sub-intervals $[a, x]$ and $[x, b]$,

$$S_{[a,x]}(k) = \begin{bmatrix} r^+_{[a,x]}(k) & t^-_{[a,x]}(k) \\ t^+_{[a,x]}(k) & r^-_{[a,x]}(k) \end{bmatrix}, \quad S_{[x,b]}(k) = \begin{bmatrix} r^+_{[x,b]}(k) & t^-_{[x,b]}(k) \\ t^+_{[x,b]}(k) & r^-_{[x,b]}(k) \end{bmatrix}. \tag{42}$$

It turns out that for each matrix there is a system of four ODEs governing its four entries. In particular, there is a scalar Riccati equation for each of the reflection coefficients $r^-_{[a,\cdot]}(k)$ and $r^+_{[\cdot,b]}(k)$, whereas there is a matrix Riccati equation for each of the two scattering matrices $S_{[a,\cdot]}(k)$ and $S_{[\cdot,b]}(k)$. The matrix Riccati equation for $S_{[a,x]}(k)$ can be obtained by merging $[0, x]$ with $[x, x+h]$, and the other Riccati equation for $S_{[\cdot,b]}(k)$ by merging $[x - h, x]$ with $[x, b]$. Since the two procedures are extremely similar, we shall only demonstrate how the first matrix Riccati equation is derived.

The following notation is required to simplify our exposition. Denote

$$S^l =: S_{[a,x]}(k), \quad S^r =: S_{[x,b]}(k), \quad S^h =: S_{[x,x+h]}(k), \tag{43}$$

so that

$$S^l = \begin{bmatrix} S^l_{11} & S^l_{12} \\ S^l_{21} & S^l_{22} \end{bmatrix}, \quad S^r = \begin{bmatrix} S^r_{11} & S^r_{12} \\ S^r_{21} & S^r_{22} \end{bmatrix}, \quad S^h = \frac{ikh}{2}\tilde{q}(x)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + O(h^2). \tag{44}$$

Furthermore, the restriction and extension operators for the two intervals $[a, x]$ and $[x, x + h]$ are

$$R^l_1 = E^l_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 + ikh \end{bmatrix} + O(h^2), \quad R^l_2 = E^l_2 = \begin{bmatrix} e^{ik(x-a)} & 0 \\ 0 & 1 \end{bmatrix}, \tag{45}$$

whereas the restriction and extension operators for the two intervals $[x-h, x]$ and $[x, x+h]$ are

$$R_1^r = E_1^r = \begin{bmatrix} 1 & 0 \\ 0 & e^{ik(b-x)} \end{bmatrix}, \quad R_2^r = E_2^r = \begin{bmatrix} 1+ikh & 0 \\ 0 & 1 \end{bmatrix} + O(h^2). \tag{46}$$

Finally, it is easy to verify that in the splitting matrix S_p

$$\begin{aligned} \begin{bmatrix} I & -J_1 S^h \\ -J_2 S^l & I \end{bmatrix}^{-1} &= \begin{bmatrix} I + J_1 S^h J_2 S^l & J_1 S^h \\ (I + J_2 S^l J_1 S^h) J_2 S^l & I + J_2 S^l J_1 S^h \end{bmatrix} + O(h^2), \\ \begin{bmatrix} I & -J_1 S^r \\ -J_2 S^h & I \end{bmatrix}^{-1} &= \begin{bmatrix} I + J_1 S^r J_2 S^h & (I + J_1 S^r J_2 S^h) J_1 S^r \\ J_2 S^h & I + J_2 S^h J_1 S^r \end{bmatrix} + O(h^2). \end{aligned}$$

The next technical lemma is a direct consequence of Theorem 4.5.

Lemma 6.1 *Denote by $S^l(x) = S^l$ and by $S^l(x+h)$ the scattering matrices for the intervals $[a, x]$ and $[a, x+h]$, respectively. Similarly, denote by $S^r(x) = S^r$ and by $S^r(x-h)$ the scattering matrices for the intervals $[x, b]$ and $[x-h, x]$, respectively. Then*

$$\begin{aligned} S^l(x+h) &= S^l(x) + ikh \begin{bmatrix} 0 & S^l_{12} \\ S^l_{21} & 2S^l_{22} \end{bmatrix} + (E_2^l + E_1^l S^l J_1) S^h (J_2 S^l R_1^l + R_2^l) + O(h^2), \\ S^r(x-h) &= S^r(x) + ikh \begin{bmatrix} 2S^r_{22} & S^r_{12} \\ S^r_{21} & 0 \end{bmatrix} + (E_1^r + E_2^r S^r J_2) S^h (J_1 S^r R_2^r + R_1^r) + O(h^2). \end{aligned}$$

Taking the limit $h \to 0$ in the preceding lemma, we obtain

Theorem 6.2 *(**Riccati equations for scattering matrices**) Denote by $S^l(x) = S^l$ and by $S^r(x) = S^r$ the scattering matrices for the intervals $[a, x]$ and $[x, b]$, respectively. Then*

$$\frac{dS^l}{dx} = \frac{ik}{2} \left\{ \tilde{q}(x)(E_2^l + S^l J_1) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (J_1^* S^l + E_2^l) + \begin{bmatrix} 0 & 2S^l_{12} \\ 2S^l_{21} & 4S^l_{22} \end{bmatrix} \right\}, \tag{47}$$

$$\frac{dS^r}{dx} = -\frac{ik}{2} \left\{ \tilde{q}(x)(E_1^r + S^r J_1^*) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (J_1 S^r + E_1^r) + \begin{bmatrix} 4S^r_{22} & 2S^r_{12} \\ 2S^r_{21} & 0 \end{bmatrix} \right\}. \tag{48}$$

Due to the sparse structure of the translation matrix J_1, the matrix Riccati equations reduce to scalar ODEs for the individual reflection and transmission coefficients for the two chunks $[a, x]$ and $[x, b]$–the four ODEs for each chunk can be solved individually, not as a simultaneous system.

Corollary 6.3 *Denote by $S^l(x) = S^l$ and by $S^r(x) = S^r$ the scattering matrices for the intervals $[a, x]$ and $[x, b]$, respectively. Then*

$$\frac{dS^l_{22}}{dx} = \frac{ik}{2} [\tilde{q}(x)(1 + S^l_{22})^2 + 4S^l_{22}], \tag{49}$$

$$\begin{aligned}\frac{dS^l_{12}}{dx} &= \frac{ik}{2}[\tilde{q}(x)(1+S^l_{22})(e^{ik(x-a)}+S^l_{12})+2S^l_{12}], &(50)\\ \frac{dS^l_{21}}{dx} &= \frac{ik}{2}[\tilde{q}(x)(1+S^l_{22})(e^{ik(x-a)}+S^l_{21})+2S^l_{21}], &(51)\\ \frac{dS^l_{11}}{dx} &= \frac{ik}{2}\tilde{q}(x)(e^{ik(x-a)}+S^l_{12})(e^{ik(x-a)}+S^l_{21}); &(52)\end{aligned}$$

and

$$\begin{aligned}\frac{dS^r_{11}}{dx} &= -\frac{ik}{2}[\tilde{q}(x)(1+S^r_{11})^2+4S^r_{11}], &(53)\\ \frac{dS^r_{12}}{dx} &= -\frac{ik}{2}[\tilde{q}(x)(1+S^r_{11})(e^{ik(b-x)}+S^r_{12})+2S^r_{12}], &(54)\\ \frac{dS^r_{21}}{dx} &= -\frac{ik}{2}[\tilde{q}(x)(1+S^r_{11})(e^{ik(b-x)}+S^r_{21})+2S^r_{21}], &(55)\\ \frac{dS^r_{22}}{dx} &= -\frac{ik}{2}\tilde{q}(x)(e^{ik(b-x)}+S^r_{12})(e^{ik(b-x)}+S^r_{21}). &(56)\end{aligned}$$

7 Initial values and independent parameters

We will again assume that $[a,b]$ is the support of the scatterer, although our discussion is valid for arbitrary a, b. We will further assume that the scattering matrix $S = S_{[a,b]}(k)$ (see (19)) for the entire interval $[a,b]$ is given as the scattering data. Since $S_{12} = t^-_{[a,b]}(k)$ is identical to $S_{21} = t^+_{[a,b]}(k)$ (see Lemma 7.1), there are only three independent parameters as opposed to four in S as the scattering data: two reflection coefficients and one transmission coefficient (see Remark 3.2)

$$S_{11} = r^+_{[a,b]}(k), \quad S_{22} = r^-_{[a,b]}(k), \quad S_{12} = t^-_{[a,b]}(k). \tag{57}$$

We now examine the four equations for the scattering matrix S^l of interval $[a,x]$. Obviously,

$$S^l(a) = S^r(b) = 0, \quad S^l(b) = S^r(a). \tag{58}$$

It follows from the uniqueness of solution of ODEs (49)–(52) that

$$S^l_{12}(x) = S^l_{21}(x), \quad x \in [a,b], \tag{59}$$

because they satisfies the same equation with the same initial value. Thus,

Lemma 7.1 *The two transmission coefficients for an arbitrary interval are identical. In other words, the two off-diagonal entries of a scattering matrix are the same.*

Consequently, there are three equations left for the scattering matrix S^l (see (49)–(52))

$$\begin{aligned}\frac{dS^l_{22}}{dx} &= \frac{ik}{2}[\tilde{q}(x)(1+S^l_{22})^2+4S^l_{22}], &(60)\\ \frac{dS^l_{12}}{dx} &= \frac{ik}{2}[\tilde{q}(x)(1+S^l_{22})(e^{ik(x-a)}+S^l_{12})+2S^l_{12}], &(61)\\ \frac{dS^l_{11}}{dx} &= \frac{ik}{2}\tilde{q}(x)(e^{ik(x-a)}+S^l_{12})^2. &(62)\end{aligned}$$

Similarly for the scattering matrix S^r, we have the initial conditions

$$S^r_{11} = S_{11}, \quad S^r_{12} = S_{12}, \quad S^r_{22} = S_{22}, \quad x = a, \tag{63}$$

for the three equations ($S^r_{21} = S^r_{12}$ due to Lemma 7.1)

$$\frac{dS^r_{11}}{dx} = -\frac{ik}{2}[\tilde{q}(x)(1 + S^r_{11})^2 + 4S^r_{11}], \tag{64}$$

$$\frac{dS^r_{12}}{dx} = -\frac{ik}{2}[\tilde{q}(x)(1 + S^r_{11})(e^{ik(b-x)} + S^r_{12}) + 2S^r_{12}], \tag{65}$$

$$\frac{dS^r_{22}}{dx} = -\frac{ik}{2}\tilde{q}(x)(e^{ik(b-x)} + S^r_{12})^2. \tag{66}$$

We can always merge S^l and S^r to obtain S, a procedure which constitutes three equations (the merging equations, see (68)). Given S, S^l can be uniquely determined by S^r; therefore, the number of independent parameters in two scattering matrices S^l and S^r is three; see Lemma 7.3 for the actual expression of S^l in terms of S^r and S. The following lemma is a reformulation of (38) for the case $S_1 = S^l$, $S_2 = S^r$ and $c = x$ with $E_1 = R_1$, $E_2 = R_2$ given by (31).

Lemma 7.2 *Denote by* S, $S^l(x) = S^l$ *and by* $S^r(x) = S^r$ *the scattering matrices for the intervals* $[a, b]$, $[a, x]$ *and* $[x, b]$, *respectively. Then*

$$S = E_1 S^l R_1 + (E_2 + E_1 S^l J_1)\left\{S^r + \frac{S^l_{22}}{1 - S^r_{11} S^l_{22}}\begin{bmatrix} S^r_{11} \\ S^r_{21} \end{bmatrix}\begin{bmatrix} S^r_{11} & S^r_{12} \end{bmatrix}\right\}(J_2 S^l R_1 + R_2). \tag{67}$$

or equivalently by the substitution $R_1 = E_1$, $R_2 = E_2$, $J_2 = J_1^*$, *and* $S^r_{21} = S^r_{12}$,

$$S = E_1 S^l E_1 + (E_2 + E_1 S^l J_1)\left\{S^r + \frac{S^l_{22}}{1 - S^r_{11} S^l_{22}}\begin{bmatrix} S^r_{11} \\ S^r_{12} \end{bmatrix}\begin{bmatrix} S^r_{11} & S^r_{12} \end{bmatrix}\right\}(J_1^* S^l E_1 + E_2). \tag{68}$$

Finally by symmetry of two scattering matrices, we equivalently have

$$S = E_2 S^r E_2 + (E_1 + E_2 S^r J_2)\left\{S^l + \frac{S^r_{11}}{1 - S^r_{11} S^l_{22}}\begin{bmatrix} S^l_{12} \\ S^l_{22} \end{bmatrix}\begin{bmatrix} S^l_{12} & S^l_{22} \end{bmatrix}\right\}(J_2^* S^r E_2 + E_1). \tag{69}$$

Proof. The main purpose of furnishing a sketch of the proof is to write down several formulae for the inverse of the 4-by-4 matrix required in the splitting matrix S_p. Firstly,

$$\begin{aligned} B &=: (I - J_2 S^l J_1 S^r)^{-1} = \left\{I - \begin{bmatrix} S^l_{22} \\ 0 \end{bmatrix}\begin{bmatrix} S^r_{11} & S^r_{12} \end{bmatrix}\right\}^{-1} \\ &= I + \frac{1}{1 - S^r_{11} S^l_{22}}\begin{bmatrix} S^l_{22} \\ 0 \end{bmatrix}\begin{bmatrix} S^r_{11} & S^r_{12} \end{bmatrix}. \end{aligned} \tag{70}$$

Secondly, as is required by the splitting matrix S_p,

$$C =: \begin{bmatrix} I & -J_1 S^r \\ -J_2 S^l & I \end{bmatrix}^{-1} = \begin{bmatrix} I & J_1 S^r \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} I & 0 \\ J_2 S^l & I \end{bmatrix}. \tag{71}$$

Then it follows from Theorem 4.5 that

$$S = \left[E_1 S^l \;\; E_2 S^r\right] \cdot C \cdot \left[\begin{array}{c} R_1 \\ R_2 \end{array}\right], \tag{72}$$

from which follows the lemma. □

Lemma 7.3 *Under the notation of the proceeding lemma and for given S and S^r, the equation (69) for S^l can be solved to yield the explicit expressions*

$$\begin{aligned}
S^l_{22} &= \frac{S_{22} - S^r_{22}}{(S^r_{12} + e^{ik(b-x)})^2 + S^r_{11}(S_{22} - S^r_{22})}, & (73)\\
S^l_{12} &= \frac{1 - S^r_{11}S^l_{22}}{(S^r_{12} + e^{ik(b-x)})^2}\left[(S^r_{12} + e^{ik(b-x)})(S_{12} - e^{ik(x-a)}S^r_{12}) - S^r_{11}(S_{22} - S^r_{22})e^{ik(x-a)}\right]\\
&= \frac{(S^r_{12} + e^{ik(b-x)})(S_{12} - e^{ik(x-a)}S^r_{12}) + S^r_{11}(S^r_{22} - S_{22})e^{ik(x-a)}}{(S^r_{12} + e^{ik(b-x)})^2 + S^r_{11}(S_{22} - S^r_{22})}, & (74)\\
S^l_{11} &= S_{11} - (S^l_{12} + e^{ik(x-a)})^2 \frac{S^r_{11}}{1 - S^r_{11}S^l_{22}}. & (75)
\end{aligned}$$

Likewise, for given S and S^l, the equation (68) for S^r can be solved to yield the explicit expressions

$$\begin{aligned}
S^r_{11} &= \frac{S_{11} - S^l_{11}}{(S^l_{12} + e^{ik(x-a)})^2 - S^l_{22}(S^l_{11} - S_{11})}, & (76)\\
S^r_{12} &= \frac{1 - S^l_{22}S^r_{11}}{(S^l_{12} + e^{ik(x-a)})^2}\left[(S^l_{12} + e^{ik(x-a)})(S_{12} - e^{ik(b-x)}S^l_{12}) - S^l_{22}(S_{11} - S^l_{11})e^{ik(b-x)}\right]\\
&= \frac{(S^l_{12} + e^{ik(x-a)})(S_{12} - e^{ik(b-x)}S^l_{12}) + S^l_{22}(S^l_{11} - S_{11})e^{ik(b-x)}}{(S^l_{12} + e^{ik(x-a)})^2 - S^l_{22}(S^l_{11} - S_{11})}, & (77)\\
S^r_{22} &= S_{22} - (S^l_{12} + e^{ik(b-x)})^2 \frac{S^l_{22}}{1 - S^l_{22}S^r_{11}}. & (78)
\end{aligned}$$

8 Other operations on the scattering matrices

In this section, we include three operations for the analytical and numerical treatment of the scattering problem. We first observe that S^r_{12} and S^r_{22} can be scaled for more efficient numerical solution. We then specify the formulae connecting the classical scattering matrix with the one used in this paper. Finally, we restate a result by Jaulent and Jean [2] which relates the scattering matrices of dissipative and active media.

8.1 Further scaling of the scattering matrices

The scattering matrix S^r is not well-scaled in the sense that S^r_{12} and S^r_{22} are highly oscillatory functions of x for large k. Let the symmetric matrix W^r be defined by

$$W^r_{11} = S^r_{11}, \tag{79}$$

$$
\begin{aligned}
W^r_{12} &= e^{ik(x-b)} S^r_{12}, && (80)\\
W^r_{22} &= e^{2ik(x-b)} S^r_{22}. && (81)
\end{aligned}
$$

The highly oscillatory modes are much less pronounced in W^r if $\tilde{q}$ is smooth in $\mathbb{R}^1$. This makes it desirable for numerical treatment.

Lemma 8.1 *Under the notation of the proceeding lemma, the symmetric matrix* W^r *is a solution of the equations*

$$
\begin{aligned}
\frac{dW^r_{11}}{dx} &= -\frac{ik}{2}[\tilde{q}(x)(1+W^r_{11})^2 + 4W^r_{11}], && (82)\\
\frac{dW^r_{12}}{dx} &= ikW^r_{12} - \frac{ik}{2}[\tilde{q}(x)(1+W^r_{11})(1+W^r_{12}) + 2W^r_{12}], && (83)\\
\frac{dW^r_{22}}{dx} &= 2ikW^r_{22} - \frac{ik}{2}\tilde{q}(x)(1+W^r_{12})^2. && (84)
\end{aligned}
$$

8.2 Link to classical definition of scattering matrix

In the remainder of the paper, we will consider the Helmholtz equation (1) in interval $[a, b]$ with $\tilde{q}(x) = q(x) \pm \frac{i}{k}m(x)$; namely, the scattering problem with dissipative and active media. We will denote by $\phi^{\pm}(x, k)$ their respective scattering solutions of the form (3), and by $S^{\pm}$ their respective scattering matrices in $[a, b]$. Following the standard definition, the classical scattering matrix $S^{\pm}_c$ is given by the formula

$$
S^{\pm}_c = \begin{bmatrix} L^{\pm} & T^{\pm} \\ T^{\pm} & R^{\pm} \end{bmatrix} \tag{85}
$$

where the coefficients $L^{\pm}$, $T^{\pm}$, and $R^{\pm}$ are related to the scattering solutions, or the Jost solutions, by the formulae

$$
\begin{aligned}
\phi^{\pm}_l(x,k) &= \begin{cases} e^{ikx} + L^{\pm}(k)e^{-ikx}, & x \le a, \\ T^{\pm}(k)e^{ikx}, & x \ge b, \end{cases} && (86)\\
\phi^{\pm}_r(x,k) &= \begin{cases} T^{\pm}(k)e^{-ikx}, & x \le a, \\ e^{-ikx} + R^{\pm}(k)e^{ikx}, & x \ge b. \end{cases} && (87)
\end{aligned}
$$

Lemma 8.2 *Suppose that* $S^{\pm}_c$ *are the classical scattering matrices defined by formula (85), that* $S^{\pm}$ *are the scattering matrices defined by formula (19), of the same scatterer* $\tilde{q}(x) = q(x) \pm \frac{i}{k}m(x)$ *and in the same interval* $[a, b]$. *Then*

$$
\begin{aligned}
L^{\pm} = e^{2ikx} \cdot S^{\pm}_{11},\ R^{\pm} = e^{-2ikb} \cdot S^{\pm}_{22},\ T^{\pm} = 1 + e^{ik(x-b)} \cdot S^{\pm}_{12}; && (88)\\
S^{+}_c(k)[S^{-}_c(k)]^* = I, \quad k \in \mathbb{R}^1. && (89)
\end{aligned}
$$

Proof. (88) follows directly from the definitions of the scattering matrices. To establish (89), we need to note that for real k (i) a flip of sign of k changes a scattering solution $u(x, k)$ of the active medium $\tilde{q}(x) = q(x) - \frac{i}{k}m(x)$ to that of the dissipative medium $\tilde{q}(x) = q(x) + \frac{i}{k}m(x)$ (ii) a Wronskian of (1) is a constant independent of $x \in \mathbb{R}^1$. The identity (89), which contains four equations, follows immediately from the evaluation at $x = a$ and $x = b$ of each of the Wronskians corresponding to the four pairs of scattering solutions $[\phi_l^+(\cdot,k),\phi_l^-(\cdot,-k)]$, $[\phi_l^+(\cdot,k),\phi_r^-(\cdot,-k)]$, $[\phi_r^+(\cdot,k),\phi_l^-(\cdot,-k)]$, and $[\phi_r^+(\cdot,k),\phi_r^-(\cdot,-k)]$. $\square$

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