Inverse scattering for lossy medium material

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Abstract. We address the problem of the recovery of coefficients, the index of refraction $n(x)$ and the dissipative $m(x)$, for the equation

$$
\phi''(x) + k^2 n(x) \phi(x) + ik \cdot m(x) \phi(x) = 0, \quad x \in \mathbb{R},
$$

and present an accurate, efficient, and stable numerical method for the reconstruction of $m(x)$ and $n(x)$ from the scattered measurements.
Plan of the Talk

- Background and applications
- Introduction to 1-D inverse scattering problems
- Existing results and our new results
- Analytical tools for our inverse problem
- The inversion algorithm
- Numerical performance
- Discussion
Background and Applications

\[ \phi''(x) + k^2 n(x) \phi(x) + i k \cdot m(x) \phi(x) = 0, \quad x \in \mathbb{R} \]

- Essentially a Helmholtz equation (1-D)

- Propagation of the time-harmonic waves through inhomogeneous medium located inside \([a, b]\). Outside \([a, b]\), \(n = 1\) and \(m = 0\).

- Reducible from Maxwell’s equation in a 3-D layered medium.

- Maxwell’s case: \(n(x)\) is permittivity, \(m(x)\) is conductivity.

- Dissipative: \(m(x) \geq 0\), solutions decay.

- Active material \(m(x) \leq 0\), solutions grow. For negative \(m\), \(m < 0\) could be quite large; ask Enrico Fermi and D. D. Joseph about their applications.
Introduction to 1-D Inverse Scattering Problems

- Inverse problem: Given solutions of differential equation $u(x, t) = u(x, t; a, b)$ on the interval $[a, b]$, determine the coefficients of the equation.

- Inverse scattering is the inverse problem for wave equation.

- A general 1-D wave equation as a motivation:

$$u_{tt}(x, t) + \beta(x)u_t(x, t) = c^2(x)\rho(x) \left[ \frac{1}{\rho(x)} u_x \right]_x, \quad c(x) = \frac{\beta(x)}{\rho(x)}$$

- By the time-harmonic substitution $u(x, t) = \phi(x)e^{-ikt}$

$$\phi''(x) + \ell(x)\phi'(x) + k^2 n(x) \phi(x) + ik \cdot m(x) \phi(x) = 0$$

with three real coefficients

$$\ell(x) = -\frac{\rho'(x)}{\rho(x)}, \quad n(x) = \frac{c_0^2}{c^2(x)}, \quad m(x) = \frac{\beta}{c^2(x)}$$
1-D Inverse Scattering Problems

- A simple Helmholtz equation (self-adjoint), already studied:
  \[ \phi''(x) + k^2 n(x) \phi(x) = 0 \]

- More complicated – two coefficients, and non-self-adjoint:
  \[ \phi''(x) + k^2 n(x) \phi(x) + ik \cdot m(x) \phi(x) = 0 \]
  \[ \rightarrow \text{the subject of this talk} \]

- Still more complicated – three coefficients, yet to be discussed:
  \[ \phi''(x) + \ell(x) \phi'(x) + k^2 n(x) \phi(x) + ik \cdot m(x) \phi(x) = 0 \]
- A simple 1-D forward scattering problem

\[ \phi''(x) + k^2 n(x) \phi(x) = 0, \quad x \in [a, b] \]

- \( k \) – wave number, a positive number in \((0, \infty)\)
- \( n \) – index of refraction, \( n(x) = 1 + q(x), \quad q = 0 \) or \( q \) is a function of \( x \)
- \( \phi \) – total wave field, \( \phi(x) = \phi_0(x) + \psi(x) \),

- Only two possible incident wave fields: \( \phi_0(x, k) = e^{ikx} \)
  \( \leadsto \) two corresponding scattered fields: \( \psi_{\pm}(x, k) \), satisfy

\[ \psi''(x) + k^2 (1 + q(x)) \psi(x) = -k^2 q(x) \phi_0(x), \quad x \in [a, b] \]

subject to the outgoing radiation conditions

\[ \psi'(a) + ik \psi(a) = 0, \quad \psi'(b) - ik \psi(b) = 0 \quad (a \text{ third} \)
Forward and Inverse Scattering Problems

\[
\psi''(x, k) + k^2(1 + q(x))\psi(x) = -k^2q(x)\phi_0(x, k),
\]
\[
\psi'(a) + ik\psi(a) = 0, \quad \psi'(b) - ik\psi(b) = 0.
\]

- Forward problem: Given \( k, q, \phi_0(x, k) \), determine \( \psi(x) \).
  \( \rightarrow \) the forward problem is well-posed.

- Inverse problem: Given \( \{ \psi_\pm(a, k), \psi_\pm(b, k), k \in (0, \infty) \} \)
  \( \{ q(x), x \in [a, b] \} \)
  \( \rightarrow \) the inverse problem is also well-posed (John Sylvester).

Remark: Only one of the two functions—the two reflection coefficients \( \psi_+(a, k), \psi_-(b, k) \)—is required to recover the scatterer.

Generalization: The scatterer \( q \) may have imaginary
Scattering Data and Scattering Matrices

- For the general, three-coefficient, equation

\[ \psi''(x, k) + \ell(x)\psi' + k^2(1 + q(x)\psi + ik \cdot m(x)\psi) = 0 \]

all four measurements \( \{ \psi_\pm(a, k), \psi_\pm(b, k), \quad k \in (0, \infty) \} \) used to recover the three coefficients \( \{ \ell(x), q(x), m(x) \} \).

- Scattering matrix: organize the four functions and study their algebraic and analytic properties for scattering problems:

\[ S(a, b, k) = \begin{bmatrix} \psi_+(a, k) & \psi_-(a, k) \\ \psi_+(b, k) & \psi_-(b, k) \end{bmatrix} \]

- A better definition for a proper scaling

\[ S(a, b, k) = \begin{bmatrix} \psi_+(a, k)e^{ika} & \psi_-(a, k)e^{ika} \\ \psi_+(b, k)e^{-ika} & \psi_-(b, k)e^{-ika} \end{bmatrix} =: \begin{bmatrix} I & F \end{bmatrix} \]

- Inverse scattering: \( \{ S(a, b, k), \quad x \in (0, \infty) \} \longrightarrow \{ \ell, q, m \} \)
- Trace formula method for inversion: construct a system

\[ S'(x, b, k) = F_0(S, k, \ell, q, m), \text{ for all } k \]

\[ \ell'(x) = F_1(\int S \, dk; \; \ell, q, m), \]

\[ q'(x) = F_2(\int S \, dk; \; \ell, q, m), \]

\[ m'(x) = F_3(\int S \, dk; \; \ell, q, m), \]

and solve them from \( a \) to \( b \) with the initial values

\[ S(a, b, k) – \text{scattering data, and } \ell(a) = q(a) = m(a). \]

- Frequency-global, space-local; ODEs amount to linear
Existing Results and Our New Results

- The simple inverse scattering problem

\[ \psi''(x) + k^2(1 + q(x))\psi(x) = -k^2q(x)\phi_0(x) \]

Chen and Rokhlin (1991): Discovered a trace formula to construct the scatterer \( q \in C^l(\mathbb{R}^1) \) from scattering data \( \{ \{ \sin kx \} , x \in (0, A]\} \) with precision \( O(1/A^l) \).

- The resulting algorithm is known as the most accurate and stable scheme.

- Technique: Use symmetry and gain super-algebraic convergence for smooth scatterer \( q \).
Existing Results and Our New Results

- The two-coefficient equation

\[
\psi''(x, k) + \ell(x)\psi' + k^2(1 + q(x)\psi + ik \cdot m(x)\psi = -1
\]

The only result (J. O. Powell, 1999) uses a first order system, and the algorithm is unstable.

- Difficulty: the equation is no longer self-adjoint; symmetry is no longer present here.

- Our results: We obtained parallel results to those of
Analytical Tools: 1. Riccati Equations

- As is well-known

  - A linear, scalar, and one-dimensional elliptic (differentiation leads to a scalar Riccati equation.

  - A linear, one-dimensional system of elliptic equations (matrix Riccati equation).

- It turns out that the scattering matrices $S^l(x) = S(x, b, k)$ satisfy the matrix Riccati equations:

\[
\frac{dS^l}{dx} = \frac{ik}{2} \left\{ q(x)(E_2^l + S^l J_1) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (J_1^* S^l + E_2^l) + \right\}
\]

\[
\frac{dS^r}{dx} = -\frac{ik}{2} \left\{ q(x)(E_1^r + S^r J_1^*) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (J_1 S^r + E_1^r) + \right\}
\]
- Entry by entry, the equations are (with $q \leftarrow q - m/i$)

\[
\frac{dS^l_{22}}{dx} = \frac{ik}{2}[q(x)(1 + S^l_{22})^2 + 4S^l_{22}],
\]

\[
\frac{dS^l_{12}}{dx} = \frac{ik}{2}[q(x)(1 + S^l_{22})(e^{ik(x-a)} + S^l_{12}) + 2S^l_{12}],
\]

\[
\frac{dS^l_{21}}{dx} = \frac{ik}{2}[q(x)(1 + S^l_{22})(e^{ik(x-a)} + S^l_{21}) + 2S^l_{21}],
\]

\[
\frac{dS^l_{11}}{dx} = \frac{ik}{2}q(x)(e^{ik(x-a)} + S^l_{12})(e^{ik(x-a)} + S^l_{22})
\]

and

\[
\frac{dS^r_{11}}{dx} = -\frac{ik}{2}[q(x)(1 + S^r_{11})^2 + 4S^r_{11}],
\]

\[
\frac{dS^r_{12}}{dx} = -\frac{ik}{2}[q(x)(1 + S^r_{11})(e^{ik(b-x)} + S^r_{12}) + 2S^r_{12}],
\]

\[
\frac{dS^r_{21}}{dx} = -\frac{ik}{2}[q(x)(1 + S^r_{11})(e^{ik(b-x)} + S^r_{21}) + 2S^r_{21}],
\]

\[
\frac{dS^r_{22}}{dx} = -\frac{ik}{2}q(x)(e^{ik(b-x)} + S^r_{12})(e^{ik(b-x)} + S^r_{22})
\]
Analytical Tools: 2. WKBJ Expansions

- The well-posedness of the Riccati equations were established; therefore, can write down the asymptotics for large frequencies.

\[
S_{22}^l(x, k) = \frac{1 - \sqrt{n}}{1 + \sqrt{n}} + \frac{1}{2n(1 + \sqrt{n})^2} \left[ -q' + 2m\sqrt{n} \right] \frac{1}{ik} \]
\[
S_{11}^r(x, k) = \frac{1 - \sqrt{n}}{1 + \sqrt{n}} + \frac{1}{2n(1 + \sqrt{n})^2} \left[ +q' + 2m\sqrt{n} \right] \frac{1}{ik} \]

where,

\[
n(x) = 1 + q(x)\]
\[
S_{22}^{2,l} = \frac{dS_{22}^{1,l}}{dx} + m \left( 1 + S_{22}^{0,l} \right) S_{22}^{1,l} - \frac{i}{2}q \left( S_{22}^{1,l} \right) \left( 1 + S_{22}^{0,l} \right) \]
\[
\quad \quad \quad \quad \quad \quad \quad \quad + i \left[ q \left( 1 + S_{22}^{0,l} \right) + 2 \right] \]
\[
S_{11}^{2,r} = \frac{dS_{11}^{1,r}}{dx} - m \left( 1 + S_{11}^{0,r} \right) S_{11}^{1,r} + \frac{i}{2}q \left( S_{11}^{1,r} \right) \left( 1 + S_{11}^{0,r} \right) \]
\[
\quad \quad \quad \quad \quad \quad \quad \quad - i \left[ q \left( 1 + S_{11}^{0,r} \right) + 2 \right] \]
Analytical Tools: 3. Trace Formulae

- For $q(x)$:

$$q' = \frac{1}{\pi} (1 + q)(1 + \sqrt{1 + q})^2 \times \lim_{A \to +\infty} \int_{-A}^{A} (S_{22}^l(x, k) - 1) dx$$

- For $m(x)$:

$$m = \frac{1}{2} \sqrt{1 + q}(1 + \sqrt{1 + q})^2 \times \lim_{A \to +\infty} \frac{1}{2A} \int_{-A}^{A} ik(S_{22}^l(x, k) - 1) dx$$

- Neither stable nor of high order for finite $A$. 

Analytical Tools: 4. Active Material

- Want to create symmetry by using some other equation
  \[\psi''(x, k) + \ell(x)\psi' + k^2(1 + q(x)\psi + ik \cdot m(x)\psi = -\psi''(x, k) + \ell(x)\psi' + k^2(1 + q(x)\psi - ik \cdot m(x)\psi = -\psi\]

- There were two scattering matrices \(S^l(x) = S(a, x, k)\) and \(S(x, b, k)\)

- There are now four scattering matrices \(S^{\pm l}(x), S^{\pm r}(x)\) contains a WKBJ expansion:
  \[
  S^{\pm l}_{22}(x, k) = \frac{1 - \sqrt{n}}{1 + \sqrt{n}} + \frac{1}{2n(1 + \sqrt{n})^2} \left[-q' \pm 2m\sqrt{n}\right] \frac{1}{ik} \\
  S^{\pm r}_{11}(x, k) = \frac{1 - \sqrt{n}}{1 + \sqrt{n}} + \frac{1}{2n(1 + \sqrt{n})^2} \left[+q' \pm 2m\sqrt{n}\right] \frac{1}{ik}
  \]
where,

\[ n(x) = 1 + q(x) \]
\[ S_{22}^{2,l} = \frac{dS_{22}^{1,l}}{dx} \pm m \left( 1 + S_{22}^{0,l} \right) S_{22}^{1,l} - \frac{i}{2} q \left( S_{22}^{1,l} \right) \left[ q \left( 1 + S_{22}^{0,l} \right) + 2 \right] \]
\[ S_{11}^{2,r} = \frac{dS_{11}^{1,r}}{dx} = m \left( 1 + S_{11}^{0,r} \right) S_{11}^{1,r} + \frac{i}{2} q \left( S_{11}^{1,r} \right) \left[ q \left( 1 + S_{11}^{0,r} \right) + 2 \right] \]

- Symmetry: It is easy to show that \( S_{22}^{+l} + S_{22}^{-l} \) and the conjugate of \( S_{11}^{+r} - S_{11}^{-r} \) have identical WKBJ expansions, with \( n \) and \( m \) on the line \( \mathbb{R}^1 \), just as in the classical case. Furthermore, \( S_{22}^{+l} = S_{22}^{-l} = S_{22}^{l} \), and \( S_{11}^{+r} = S_{11}^{-r} = S_{11}^{r} \) where the reflection coefficients satisfy \( S_{22}^{l} = S_{11}^{r} \).
Analytical Tools: 4. New Trace Formulae

The newly established symmetry can be used in a simple way to yield trace formulae:

- For $q(x)$,
  \[ q' = \frac{1}{\pi} \frac{1}{1+q} \left(1+\sqrt{1+q}\right)^2 \times \int_{-\infty}^{\infty} (S_{22}^{+l} + S_{22}^{-l} - S_{11}^{+r} - S_{22}^{-r}) \]

- For $m(x)$,
  \[ m = \frac{1}{2} \frac{1}{\sqrt{1+q}} \left(1+\sqrt{1+q}\right)^2 \times \int_{-\infty}^{\infty} (S_{22}^{+l} + S_{11}^{+r} - S_{22}^{-l} - S_{22}^{-r}) \]

- Stable and super-algebraic convergent
Analytical Tools: 5. Merging and Conjugate Operations

- Remember only the scattering matrix $S^{+r}(a, k)$ is used for the scattering data, which is to be used as the initial value for the equation for $S^{+r}(x, k)$.

- Fortunately, the other three scattering matrices $S^{+l}$, $S^{−l}$, and $S^{−r}$ can be obtained with the merging and conjugate operations.

- Merging is useful to calculate the scattering matrix $S^{+r}$ from the right chunk $[a, x]$ from that for the right chunk $[x, b]$; namely, $S^{+l}$ from $S^{+r}$, and $S^{−l}$ from $S^{−r}$.

- The conjugate operation is useful to obtain the scattering matrices for the active material from the those of the passive material.
Analytical Tools: 5.1. Merging Operation

- The vector form of the formula:

\[
S = E_2 S^r E_2 + (E_1 + E_2 S^r J_2) \left\{ S^l + \frac{S^r_{11}}{1-S^r_{11} S^r_{22}} \begin{bmatrix} S^l_{12} \\ S^l_{22} \end{bmatrix} \begin{bmatrix} S^l_{11} & S^l_{22} \end{bmatrix} \right\}
\]

where \( E_1, E_2, J_2 \) are some 2-by-2 constant matrices

- The explicit, scalar form of the formula:

\[
S^l_{22} = \frac{S^r_{22} - S_{22}}{S^r_{11}(S^r_{22} - S_{22}) - (S^r_{12} + e^{ik(b-x)})^2},
\]

\[
S^l_{12} = \frac{1 - S^r_{11} S^l_{22}}{(S^r_{12} + e^{ik(b-x)})^2} \times
\]

\[
[(S^r_{12} + e^{ik(b-x)})(S_{12} - e^{ik(x-a)}S^r_{12}) - S^r_{11}(S_{22} - S^r_{22})]
\]

\[
= \frac{(S^r_{12} + e^{ik(b-x)})(e^{ik(x-a)}S^r_{12} - S_{12}) - S^r_{11}(S^r_{22} - S_{22})e^{ik}}{S^r_{11}(S^r_{22} - S_{22}) - (S^r_{12} + e^{ik(b-x)})^2}
\]

\[
S^l_{11} = S_{11} - (S^l_{12} + e^{ik(x-a)})^2 \frac{S^r_{11}}{1-S^r_{11} S^l_{22}}.
\]
Analytical Tools: 5.2. Conjugate Operation

The conjugate operation is quite simple, and is a direct consequence of an examination of the Wronskians of the two equivalent passive and active media.

\[ S^{+l}(x, k) \cdot (S^{-l}(x, k))^* = I; \quad S^{+r}(x, k) \cdot (S^{-r}(x, k))^* = I \]

provided that \( S^{-l}(x, k) \) and \( S^{-r}(x, k) \) exist.

- The well-posedness of the active medium problem exists only for small \( m \).
The Inversion Algorithm

Solve the initial value problem for $q$ and $S^{+r}(x, k)$ for
and for each of the positive frequencies $k \in (0, \infty)\$

$$\frac{dS^{+r}}{dx} = -\frac{ik}{2} \left\{ (q - m/ik)(E_1^r + S^{+r} J_1^*) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (J_1 S_{11}^r + 2S_{12}^r) \\
+ \begin{bmatrix} 4S_{22}^{+r} & 2S_{12}^{+r} \\ 2S_{21}^{+r} & 0 \end{bmatrix} \right\},$$

$$q' = \frac{1}{\pi} (1 + q)(1 + \sqrt{1+q})^2 \times \int_\infty^{-\infty} (S_{22}^{+l} + S_{22}^{-l} - S_{11}^{+r})$$

$$m = \frac{1}{2} \sqrt{1+q} (1 + \sqrt{1+q})^2 \times \int_\infty^{-\infty} (S_{22}^{+l} + S_{11}^{+r} - S_{22}^{-l})$$

with the initial values $S^{+r}(a, k)$, $q(a) = m(a) = 0$.

- The required entries $S_{22}^{+l}$, $S_{22}^{-l}$, $S_{11}^{-r}$ can be obtained from the merging and conjugate operations.
Discussions and Conclusions

- It seems that no one has been able to contructed \( \frac{1}{m} \) of some intermediate order: from second order and \( \frac{1}{m} \) have first order or super-algebraic convergence.

- Owing to the use of the active material, there is a threshold in the magnitude of the dissipative term \( m \). The trace method in general will not work for large \( m \).

- Trace method is a special case of the so-called space-local, frequency-global approach. It is not a flexible method in the sense that the overwhelming analytical and algebraic requirements of functionally fitting terms in the formulae.

- Space-local, frequency-global approaches v.s. Space-local
  → different ways to linearize
- SLFG is always well-posed, easier to analyze, but difficult to discretize $x$ in high order; inefficient in utilizing the source data; unable to recover discontinuous coefficients without extrapolating.

- SGFL is ill-posed, difficult to analyze, but there is no instability in actual computation; efficient in utilizing the source data; easy to construct high order schemes; convenient to recover discontinuous coefficients.

- SGFL is the choice for accurate and reliable algorithms for inversion.

- SGFL is expected to work for the inversion of $n$ and $m$. 