Inverse scattering for lossy medium via active material

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Abstract. We address the problem of the recovery of two coefficients, the index of refraction $n(x)$ and the dissipation coefficient $m(x)$, for the equation

$$\phi''(x) + k^2 n(x)\phi(x) + ik \cdot m(x)\phi(x) = 0, \quad x \in [a, b];$$

and present an accurate, efficient, and stable numerical method for the reconstruction of m(x) and n(x) from the scattering measurements.

Plan of the Talk

- Background and applications
- Introduction to 1-D inverse scattering problems
- Existing results and our new results
- Analytical tools for our inverse problem
- The inversion algorithm
- Numerical performance
- Discussion

Background and Applications

$$\phi''(x) + k^2 n(x)\phi(x) + ik \cdot m(x)\phi(x) = 0, \quad x \in [a, b];$$

- Essentially a Helmholtz equation (1-D)

- Propagation of the time-harmonic waves through inhomogeneous medium located inside $[a, b]$. Outside $[a, b]$, $n = 1$ and $m = 0$.

- Reducible from Maxwell's equation in a 3-D layered medium

- Maxwell's case: $n(x)$ is permittivity, $m(x)$ is conductivity.

- Dissipative: $m(x) \geq 0$, solutions decay.

- Active material $m(x) \leq 0$, solutions grow. For neutron diffusion, $m < 0$ could be quite large; ask Enrico Fermi and Dr. von Brown about their applications.

Introduction to 1-D Inverse Scattering Problems

- Inverse problem: Given solutions of differential equation outside $[a, b]$, determine the coefficients of the equation.

- Inverse scattering is the inverse problem for wave equation.

- A general 1-D wave equation as a motivation:

$$u_{tt}(x,t) + \beta(x)u_t(x,t) = c^2(x)\rho(x)\left[\frac{1}{\rho(x)}u_x\right]_x, \quad c(x) = c_0, \quad x \notin [a,b]$$

- By the time-harmonic substitution $u(x,t) = \phi(x)e^{-i\omega t}$,

$$\phi''(x) + \ell(x)\phi'(x) + k^2 n(x)\phi(x) + ik \cdot m(x)\phi(x) = 0$$

with three real coefficients

$$\ell(x) = -\frac{\rho'(x)}{\rho(x)}, \quad n(x) = \frac{c_0^2}{c^2(x)}, \quad m(x) = \frac{\beta(x)}{c^2(x)}.$$

1-D Inverse Scattering Problems

- A simple Helmholtz equation (self-adjoint), already solved

$$\phi''(x) + k^2 n(x)\phi(x) = 0$$

- More complicated – two coefficients, and non-self-adjoint:

$$\phi''(x) + k^2 n(x)\phi(x) + ik \cdot m(x)\phi(x) = 0$$

$\hookrightarrow$ the subject of this talk

- Still more complicated – three coefficients, yet to be solved

$$\phi''(x) + \ell(x)\phi'(x) + k^2 n(x)\phi(x) + ik \cdot m(x)\phi(x) = 0$$

- A simple 1-D *forward* scattering problem

$$\phi''(x) + k^2 n(x)\phi(x) = 0, \quad x \in [a, b]$$

  - k – wave number, a positive number in $(0, \infty)$
  - n – index of refraction, $n(x) = 1 + q(x)$, $q = 0$ outside $[a, b]$
  - ϕ – total wave field, $\phi(x) = \phi_0(x) + \psi(x)$,

- Only two possible incident wave fields: $\phi_0(x, k) = e^{\pm ikx}$
  $\hookrightarrow$ two corresponding scattered fields: $\psi_\pm(x, k)$, satisfying

$$\psi''(x) + k^2(1 + q(x))\psi(x) = -k^2 q(x)\phi_0(x), \quad x \in [a, b]$$

  subject to the outgoing radiation conditions

$$\psi'(a) + ik\psi(a) = 0, \quad \psi'(b) - ik\psi(b) = 0 \quad \text{(a third type b.c.)}$$

Forward and Inverse Scattering Problems

$$\psi''(x,k) + k^2(1+q(x))\psi(x) = -k^2 q(x)\phi_0(x,k), \quad x \in [a,b]$$
$$\psi'(a) + ik\psi(a) = 0, \quad \psi'(b) - ik\psi(b) = 0$$

- Forward problem: Given $k, q, \phi_0(x,k)$, determine $\psi(x,k), x \in [a,b]$
  $\hookrightarrow$ the forward problem is well-posed.

- Inverse problem: Given $\{\ \psi_\pm(a,k), \psi_\pm(b,k), k \in (0,\infty)\ \}$, determine $\{\ q(x), x \in [a,b]\ \}$
  $\hookrightarrow$ the inverse problem is also well-posed (John Sylvester, 1996)

Remark: Only one of the two functions–the two reflected waves $\psi_+(a,k), \psi_-(b,k)$–is required to recover the scatterer q

Generalization: The scatterer q may have imaginary part.

Scattering Data and Scattering Matrices

- For the general, three-coefficient, equation

$$\psi''(x,k) + \ell(x)\psi' + k^2(1 + q(x)\psi + ik \cdot m(x)\psi = -k^2 q\phi_0(x,k)$$

all four measurements $\{ \psi_\pm(a,k), \psi_\pm(b,k), \quad k \in (0,\infty) \}$ must be used to recover the three coefficients $\{ \ell(x), q(x), m(x), \ x \in [a,b] \}$

- Scattering matrix: organize the four functions and exploit their algebraic and analytic properties for scattering problems

$$S(a,b,k) = \begin{bmatrix} \psi_+(a,k) & \psi_-(a,k) \\ \psi_+(b,k) & \psi_-(b,k) \end{bmatrix}$$

- A better definition for a proper scaling

$$S(a,b,k) = \begin{bmatrix} \psi_+(a,k)e^{ika} & \psi_-(a,k)e^{ika} \\ \psi_+(b,k)e^{-ikb} & \psi_-(b,k)e^{-ikb} \end{bmatrix} =: \begin{bmatrix} R_+ & T_- \\ T_+ & R_- \end{bmatrix}$$

- Inverse scattering: $\{S(a,b,k), \ x \in (0,\infty)\} \longrightarrow \{\ell, q, m, \ \ x \in [a,b]\}$

- Trace formula method for inversion: construct a system of ODEs

$$\begin{aligned} S'(x,b,k) &= F_0(S,k,\ell,q,m), \text{ for all } k \\ \ell'(x) &= F_1(\int S\ dk;\ \ell,q,m), \\ q'(x) &= F_2(\int S\ dk;\ \ell,q,m), \\ m'(x) &= F_3(\int S\ dk;\ \ell,q,m), \end{aligned}$$

and solve them from a to b with the initial values

$S(a,b,k)-$ scattering data, and $\ell(a)=q(a)=m(a)=0$.

- Frequency-global, space-local; ODEs amount to linearization

Existing Results and Our New Results

- The simple inverse scattering problem

$$\psi''(x) + k^2(1 + q(x))\psi(x) = -k^2 q(x)\phi_0(x)$$

Chen and Rokhlin (1991): Discovered a trace formula to reconstruct the scatterer $q \in C^l(\mathbb{R}^1)$ from scattering data $\{ R_+(a, b, k), k \in (0, A] \}$ with precision $O(1/A^l)$.

- The resulting algorithm is known as the most accurate, efficient, and stable scheme.

- Technique: Use symmetry and gain super-algebraic convergence for smooth scatterer q.

Existing Results and Our New Results

- The two-coefficient equation

$$\psi''(x,k) + \ell(x)\psi' + k^2(1 + q(x)\psi + ik \cdot m(x)\psi = -k^2 q\phi_0(x,k)$$

The only result (J. O. Powell, 1999) uses a first order trace formula, and the algorithm is unstable.

- Difficulty: the equation is no longer self-adjoint; symmetry is no longer present here.

- Our results: We obtained parallel results to those of 1991.

Analytical Tools: 1. Riccati Equations

- As is well-known

  - A linear, scalar, and one-dimensional elliptic (differential) equation leads to a scalar Riccati equation.
  - A linear, one-dimensional system of elliptic equations leads to a matrix Riccati equation.

- It turns out that the scattering matrices $S^l(x) = S(a, x, k)$ and $S^r(x) = S(x, b, k)$ satisfy the matrix Riccati equations (with $q \leftarrow q - m/ik$)

$$\frac{dS^l}{dx} = \frac{ik}{2}\left\{q(x)(E_2^l + S^l J_1)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}(J_1^* S^l + E_2^l) + \begin{bmatrix} 0 & 2S_{12}^l \\ 2S_{21}^l & 4S_{22}^l \end{bmatrix}\right\},$$

$$\frac{dS^r}{dx} = -\frac{ik}{2}\left\{q(x)(E_1^r + S^r J_1^*)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}(J_1 S^r + E_1^r) + \begin{bmatrix} 4S_{22}^r & 2S_{12}^r \\ 2S_{21}^r & 0 \end{bmatrix}\right\}.$$

- Entry by entry, the equations are (with $q \leftarrow q - m/ik$):

$$\frac{dS^l_{22}}{dx} = \frac{ik}{2}[q(x)(1+S^l_{22})^2 + 4S^l_{22}], \quad (1)$$

$$\frac{dS^l_{12}}{dx} = \frac{ik}{2}[q(x)(1+S^l_{22})(e^{ik(x-a)} + S^l_{12}) + 2S^l_{12}], \quad (2)$$

$$\frac{dS^l_{21}}{dx} = \frac{ik}{2}[q(x)(1+S^l_{22})(e^{ik(x-a)} + S^l_{21}) + 2S^l_{21}], \quad (3)$$

$$\frac{dS^l_{11}}{dx} = \frac{ik}{2}q(x)(e^{ik(x-a)} + S^l_{12})(e^{ik(x-a)} + S^l_{21}); \quad (4)$$

and

$$\frac{dS^r_{11}}{dx} = -\frac{ik}{2}[q(x)(1+S^r_{11})^2 + 4S^r_{11}], \quad (5)$$

$$\frac{dS^r_{12}}{dx} = -\frac{ik}{2}[q(x)(1+S^r_{11})(e^{ik(b-x)} + S^r_{12}) + 2S^r_{12}], \quad (6)$$

$$\frac{dS^r_{21}}{dx} = -\frac{ik}{2}[q(x)(1+S^r_{11})(e^{ik(b-x)} + S^r_{21}) + 2S^r_{21}], \quad (7)$$

$$\frac{dS^r_{22}}{dx} = -\frac{ik}{2}q(x)(e^{ik(b-x)} + S^r_{12})(e^{ik(b-x)} + S^r_{21}). \quad (8)$$

Analytical Tools: 2. WKBJ Expansions

- The well-posedness of the Riccati equations were established. We therefore can write down the asymptotics for large frequency k:

$$
\begin{aligned}
S_{22}^{l}(x,k) &= \frac{1-\sqrt{n}}{1+\sqrt{n}} + \frac{1}{2n(1+\sqrt{n})^2}\left[-q' + 2m\sqrt{n}\right]\frac{1}{ik} + S_{22}^{2,l}\frac{1}{k^2} + \ldots \\
S_{11}^{r}(x,k) &= \frac{1-\sqrt{n}}{1+\sqrt{n}} + \frac{1}{2n(1+\sqrt{n})^2}\left[+q' + 2m\sqrt{n}\right]\frac{1}{ik} + S_{11}^{2,r}\frac{1}{k^2} + \ldots
\end{aligned}
$$

where,

$$
\begin{aligned}
n(x) &= 1 + q(x) \\
S_{22}^{2,l} &= \frac{\frac{dS_{22}^{1,l}}{dx} + m\left(1+S_{22}^{0,l}\right)S_{22}^{1,l} - \frac{i}{2}q\left(S_{22}^{1,l}\right)^2}{i\left[q\left(1+S_{22}^{0,l}\right)+2\right]} \\
S_{11}^{2,r} &= \frac{\frac{dS_{11}^{1,r}}{dx} - m\left(1+S_{11}^{0,r}\right)S_{11}^{1,r} + \frac{i}{2}q\left(S_{11}^{1,r}\right)^2}{-i\left[q\left(1+S_{11}^{0,r}\right)+2\right]}
\end{aligned}
$$

Analytical Tools: 3. Trace Formulae

- For $q(x)$:

$$q' = \frac{1}{\pi}(1+q)(1+\sqrt{1+q})^2 \times \lim_{A\to+\infty} \int_{-A}^{A} (S_{22}^{l}(x,k) - S_{11}^{r}(x,k))\, dk$$

- For $m(x)$:

$$m = \frac{1}{2}\sqrt{1+q}\,(1+\sqrt{1+q})^2 \times \lim_{A\to+\infty} \frac{1}{2A} \int_{-A}^{A} ik(S_{22}^{l}(x,k) + S_{11}^{r}(x,k))\, dk$$

- Neither stable nor of high order for finite A.

Analytical Tools: 4. Active Material

- Want to create symmetry by using some other equations

$$\begin{aligned}\psi''(x,k)+\ell(x)\psi'+k^2(1+q(x)\psi+ik\cdot m(x)\psi &= -k^2q\phi_0(x,k)\\ \psi''(x,k)+\ell(x)\psi'+k^2(1+q(x)\psi-ik\cdot m(x)\psi &= -k^2q\phi_0(x,k)\end{aligned}$$

- There were two scattering matrices $S^l(x)=S(a,x,k)$ and $S^r(x)=S(x,b,k)$

- There are now four scattering matrices $S^{\pm l}(x)$, $S^{\pm r}(x)$, each sustains a WKBJ expansion:

$$\begin{aligned}S_{22}^{\pm l}(x,k) &= \frac{1-\sqrt{n}}{1+\sqrt{n}}+\frac{1}{2n(1+\sqrt{n})^2}\left[-q'\pm 2m\sqrt{n}\right]\frac{1}{ik}+S_{22}^{2,l}\frac{1}{k^2}+\ldots\\ S_{11}^{\pm r}(x,k) &= \frac{1-\sqrt{n}}{1+\sqrt{n}}+\frac{1}{2n(1+\sqrt{n})^2}\left[+q'\pm 2m\sqrt{n}\right]\frac{1}{ik}+S_{11}^{2,r}\frac{1}{k^2}+\ldots\end{aligned}$$

where,

$$
\begin{aligned}
n(x) &= 1 + q(x) \\
S_{22}^{2,l} &= \frac{\frac{dS_{22}^{1,l}}{dx} \pm m\left(1 + S_{22}^{0,l}\right) S_{22}^{1,l} - \frac{i}{2} q \left(S_{22}^{1,l}\right)^2}{i\left[q\left(1 + S_{22}^{0,l}\right) + 2\right]} \\
S_{11}^{2,r} &= \frac{\frac{dS_{11}^{1,r}}{dx} \mp m\left(1 + S_{11}^{0,r}\right) S_{11}^{1,r} + \frac{i}{2} q \left(S_{11}^{1,r}\right)^2}{-i\left[q\left(1 + S_{11}^{0,r}\right) + 2\right]}
\end{aligned}
$$

- Symmetry: It is easy to show that $S_{22}^{+l} + S_{22}^{-l}$ and the complex conjugate of $S_{11}^{+r} - S_{11}^{-r}$ have identical WKBJ expansions for smooth n and m on the line $\mathbb{R}^1$, just as in the classical case ($m = 0$, $S_{22}^{+l} = S_{22}^{-l} = S_{22}^{l}$, and $S_{11}^{+r} = S_{11}^{-r} = S_{11}^{r}$) where the two reflection coefficients satisfy $S_{22}^{l} = \overline{S_{11}^{r}}$.

Analytical Tools: 4. New Trace Formulae

The newly established symmetry can be used in a straightforward way to yield trace formulae:

- For $q(x)$,

$$q' = \frac{1}{\pi}(1+q)(1+\sqrt{1+q})^2 \times \int_{-\infty}^{\infty} (S_{22}^{+l} + S_{22}^{-l} - S_{11}^{+r} - S_{11}^{-r})(x,k)\, dk$$

- For $m(x)$,

$$m = \frac{1}{2}\sqrt{1+q}\,(1+\sqrt{1+q})^2 \times \int_{-\infty}^{\infty} (S_{22}^{+l} + S_{11}^{+r} - S_{22}^{-l} - S_{11}^{-r})(x,k)\, dk$$

- Stable and super-algebraic convergent

Analytical Tools: 5. Merging and Conjugate Operations

- Remember only the scattering matrix $S^{+r}(a,k)$ is given as the scattering data, which is to be used as the inital value for the Riccati equaiton for $S^{+r}(x,k)$.

- Fortunately, the other three scattering matrices S^{+l}, S^{-r}, and S^{-l} can be obtained with the merging and conjugate operations

- Merging is useful to calculate the scattering matrix for the left chunk $[a,x]$ from that for the right chunk $[x,b]$; namely, to calculate S^{+l} from S^{+r}, and S^{-l} from S^{-r}.

- The conjugate operation is useful to obtain the scattering matrices for the active material from the those of the passive one.

Analytical Tools: 5.1. Merging Operation

- The vector form of the formula:

$$S = E_2 S^r E_2 +$$
$$(E_1 + E_2 S^r J_2) \left\{ S^l + \frac{S^r_{11}}{1 - S^r_{11} S^l_{22}} \begin{bmatrix} S^l_{12} \\ S^l_{22} \end{bmatrix} \begin{bmatrix} S^l_{12} & S^l_{22} \end{bmatrix} \right\} (J_2^* S^r E_2 + E_1).$$

where $E_1,\ E_2,\ J_2$ are some 2-by-2 constant matrices.

- The explicit, scalar form of the formula:

$$S^l_{22} = \frac{S^r_{22} - S_{22}}{S^r_{11}(S^r_{22} - S_{22}) - (S^r_{12} + e^{ik(b-x)})^2},$$
$$S^l_{12} = \frac{1 - S^r_{11} S^l_{22}}{(S^r_{12} + e^{ik(b-x)})^2} \times$$
$$[(S^r_{12} + e^{ik(b-x)})(S_{12} - e^{ik(x-a)} S^r_{12}) - S^r_{11}(S_{22} - S^r_{22}) e^{ik(x-a)}]$$
$$= \frac{(S^r_{12} + e^{ik(b-x)})(e^{ik(x-a)} S^r_{12} - S_{12}) - S^r_{11}(S^r_{22} - S_{22}) e^{ik(x-a)}}{S^r_{11}(S^r_{22} - S_{22}) - (S^r_{12} + e^{ik(b-x)})^2},$$
$$S^l_{11} = S_{11} - (S^l_{12} + e^{ik(x-a)})^2 \frac{S^r_{11}}{1 - S^r_{11} S^l_{22}}.$$

Analytical Tools: 5.2. Conjugate Operation

The conjugate operation is quite simple, and is a direct result from an examination of the Wronskians of the two equations for the passive and active media.

$$S^{+l}(x,k)\cdot(S^{-l}(x,k))^{*}=I;\quad S^{+r}(x,k)\cdot(S^{-r}(x,k))^{*}=I.$$

provided that $S^{-l}(x,k)$ and $S^{-r}(x,k)$ exist.

- The well-posedness of the active medium problem is guaranteed only for small m.

The Inversion Algorithm

Solve the inital value problem for q and $S^{+r}(x,k)$ for x from a to b, and for each of the positive frequencies $k \in (0,\infty)$

$$\frac{dS^{+r}}{dx} = -\frac{ik}{2}\left\{(q - m/ik)(E_1^r + S^{+r}J_1^*)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}(J_1 S^{+r} + E_1^r) + \right.$$
$$\left. + \begin{bmatrix} 4S_{22}^{+r} & 2S_{12}^{+r} \\ 2S_{21}^{+r} & 0 \end{bmatrix}\right\},$$

$$q' = \tfrac{1}{\pi}(1+q)(1+\sqrt{1+q})^2 \times \textstyle\int_{-\infty}^{\infty}(S_{22}^{+l} + S_{22}^{-l} - S_{11}^{+r} - S_{11}^{-r})(x,k)\,dk$$

$$m = \tfrac{1}{2}\sqrt{1+q}\,(1+\sqrt{1+q})^2 \times \textstyle\int_{-\infty}^{\infty}(S_{22}^{+l} + S_{11}^{+r} - S_{22}^{-l} - S_{11}^{-r})(x,k)\,dk$$

with the initial values $S^{+r}(a,k),\ q(a) = m(a) = 0$.

- The required entries $S_{22}^{+l},\ S_{22}^{-l},\ S_{11}^{-r}$ can be obtained from S^{+r} via the merging and conjugate opertions.

Discussions and Conclusions

- It seems that no one has been able to contructed trace formulae of some intermediate order: from second order and up. We either have first order or super-algebraic convergence.

- Owing to the use of the active material, there is a restriction on the magnitude of the dissipative term m. The trace method fails to work for large m.

- Trace method is a special case of the so-called space-local, frequency-global approach. It is not a flexible method in the sense of overwhelming analytical and algebraic requirements of functions involved in the formulae.

- Space-local, frequency-global approaches v.s. Space-global, frequency-local

  $\hookrightarrow$ different ways to linearize

- SLFG is always well-posed, easier to analyze, but difficult to discretize x in high order; inefficient in utilizing the scattering data; unable to recover discontinuous coefficients without Gibbs overshooting.

- SGFL is ill-posed, difficult to analyze, but there is no real issue of instability in actual computation; efficient in utilizing the scattering data; easy to construct high order schemes; convenient in recovering discontinuous coefficients

- SGFL is the choice for accurate and reliable algorithm for 1-D inversion

- SGFL is expected to work for the inversion of n and m for large m.