Recent Results on Semialgebraic Range Searching Lower Bounds

Some Overview of Data Structure Lower Bounds

Contents:

Introduction

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Must avoid icebergs!

The Pointer Machine Model

Range Reporting

Range Reporting:

- A general class of Computational Geometric problems
- Input: A set of n objects, e.g., points, given by coordinates.
  - In 2D we have (x_i, y_i), $1 \leq i \leq n$
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- Answer queries
- A query is a geometric region or object.
  - Triangle
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- k: Output size

A 2D Range Reporting Example

- Input: n points in 2D
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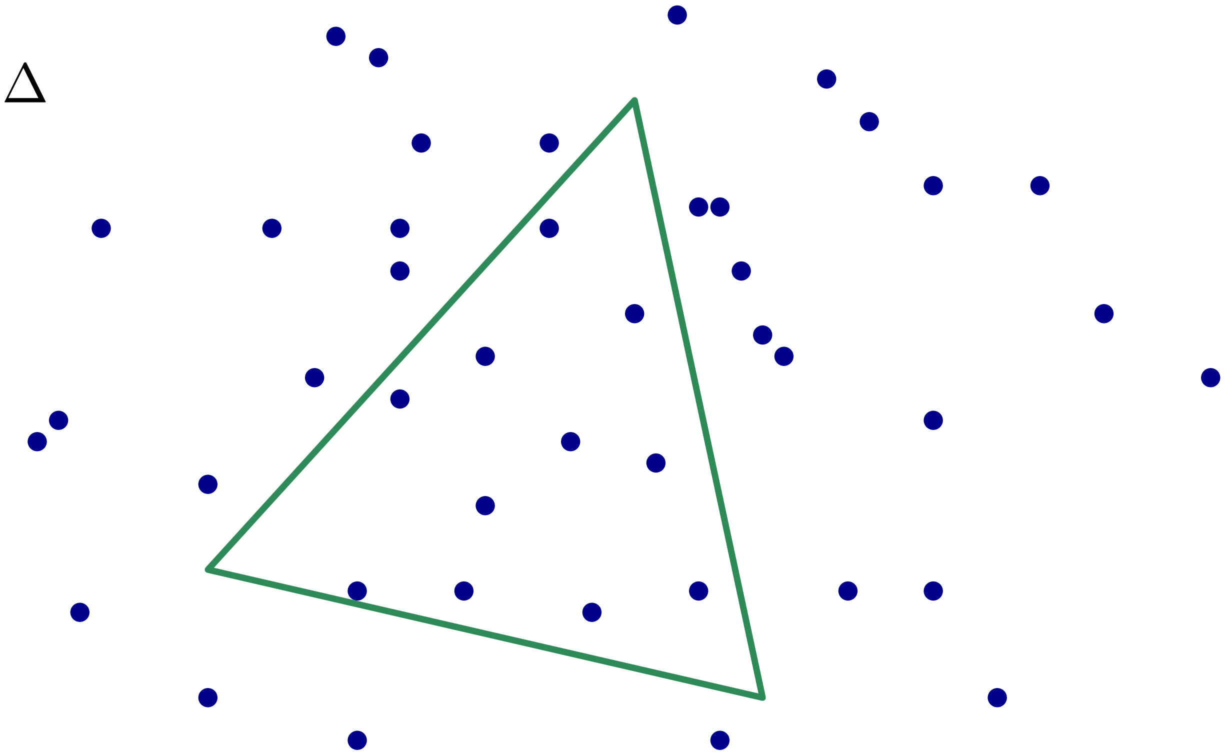
- Output: List of k points inside Δ



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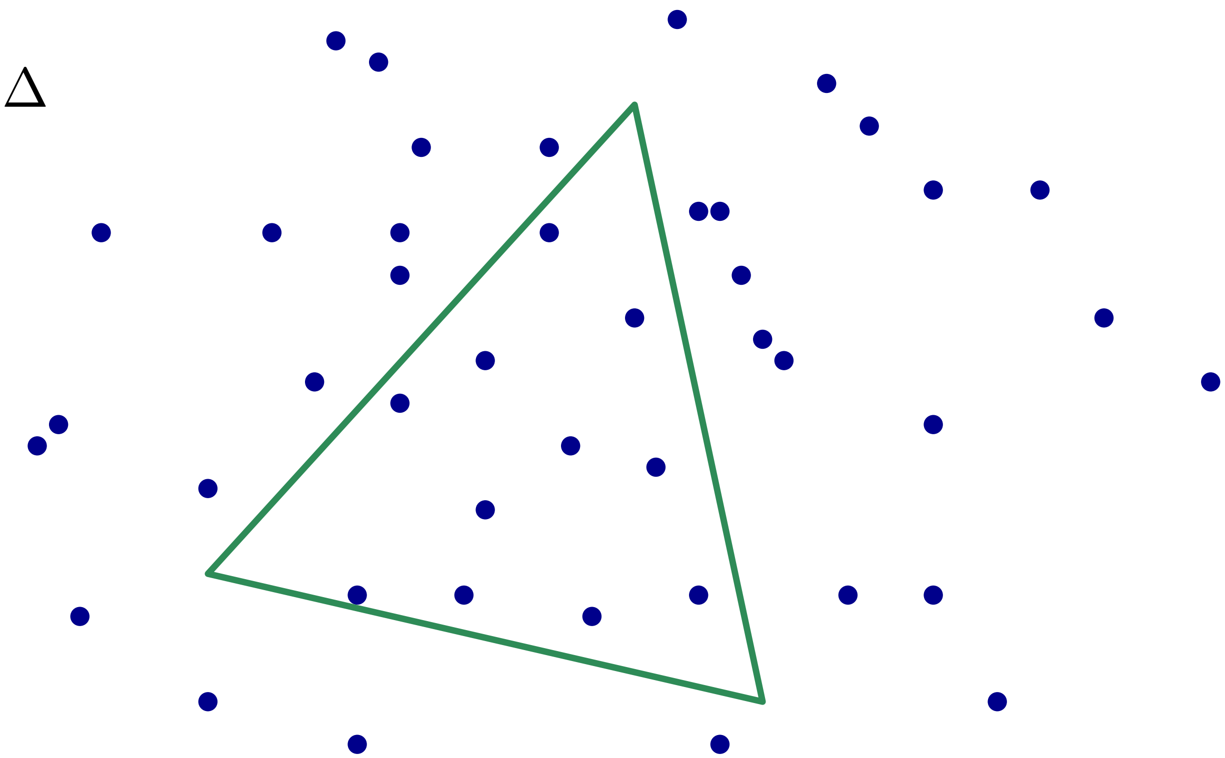

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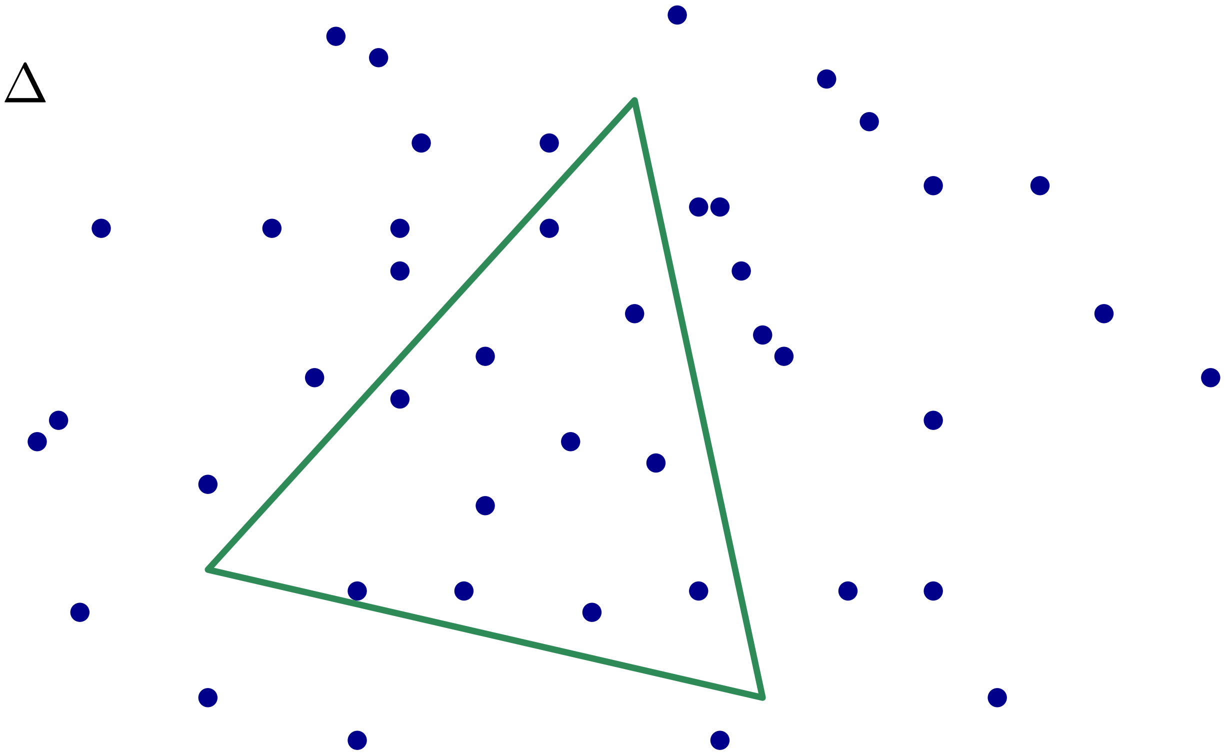

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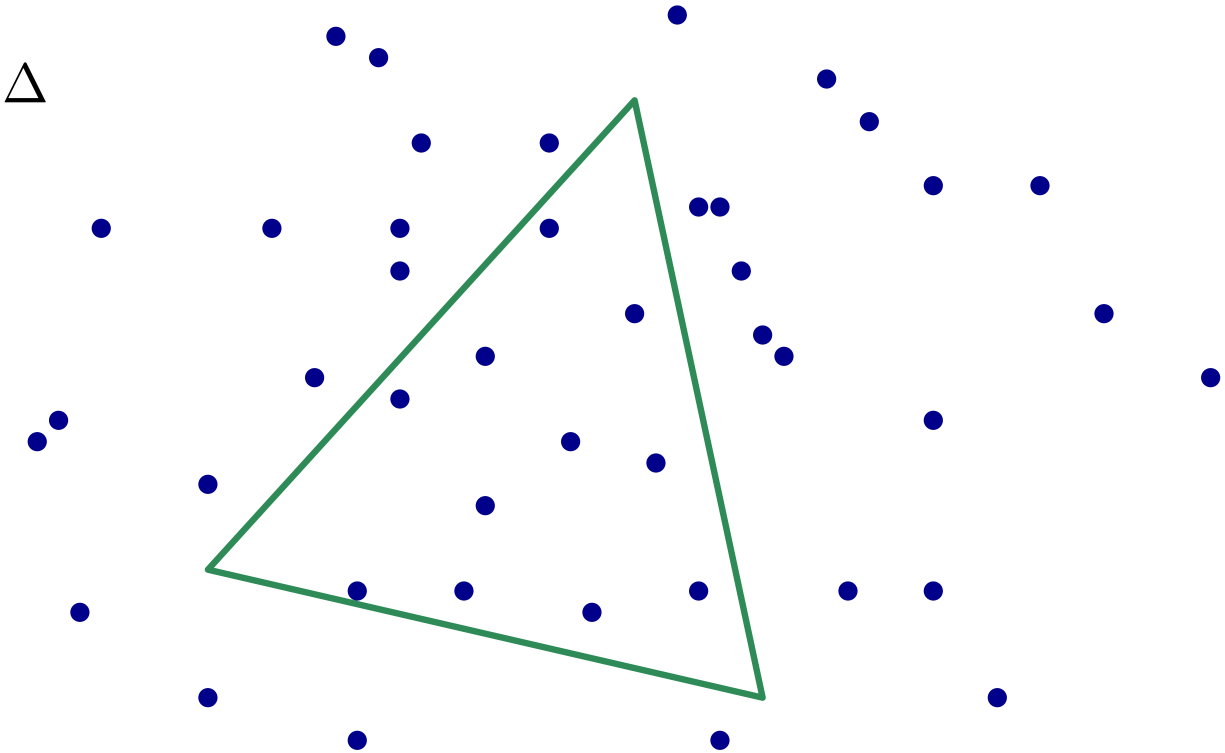

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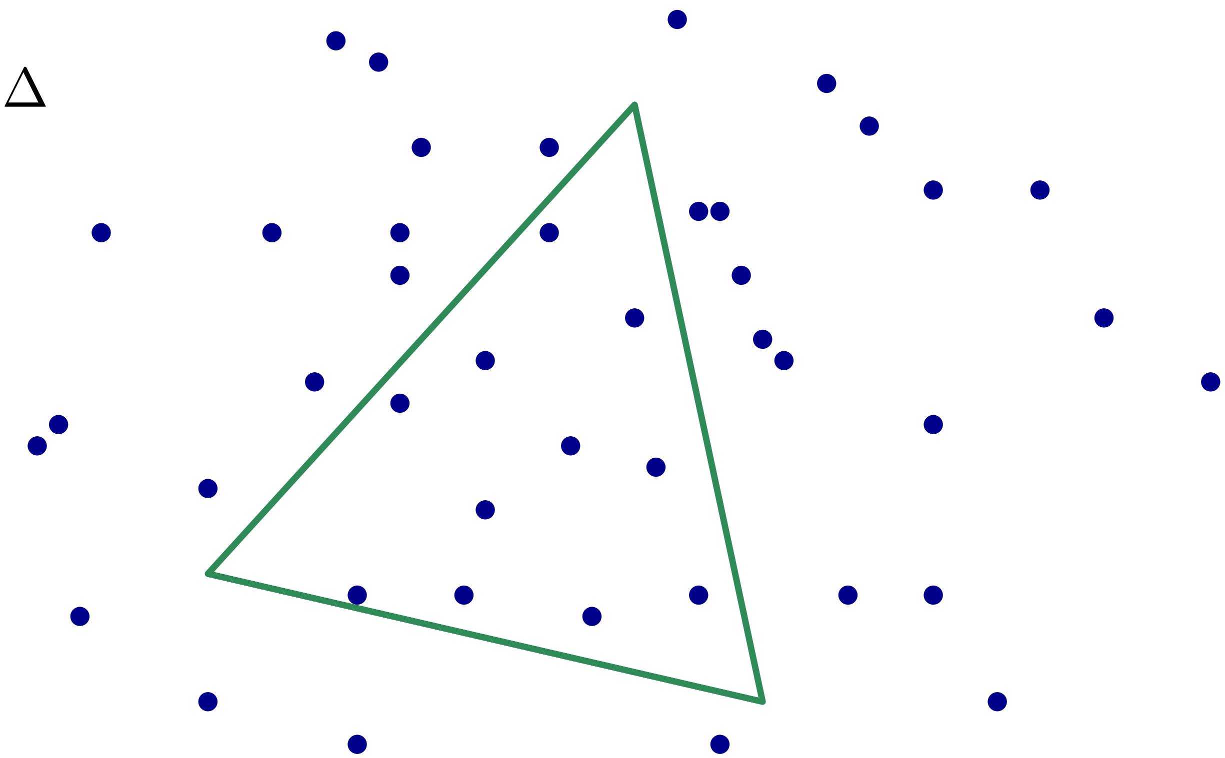

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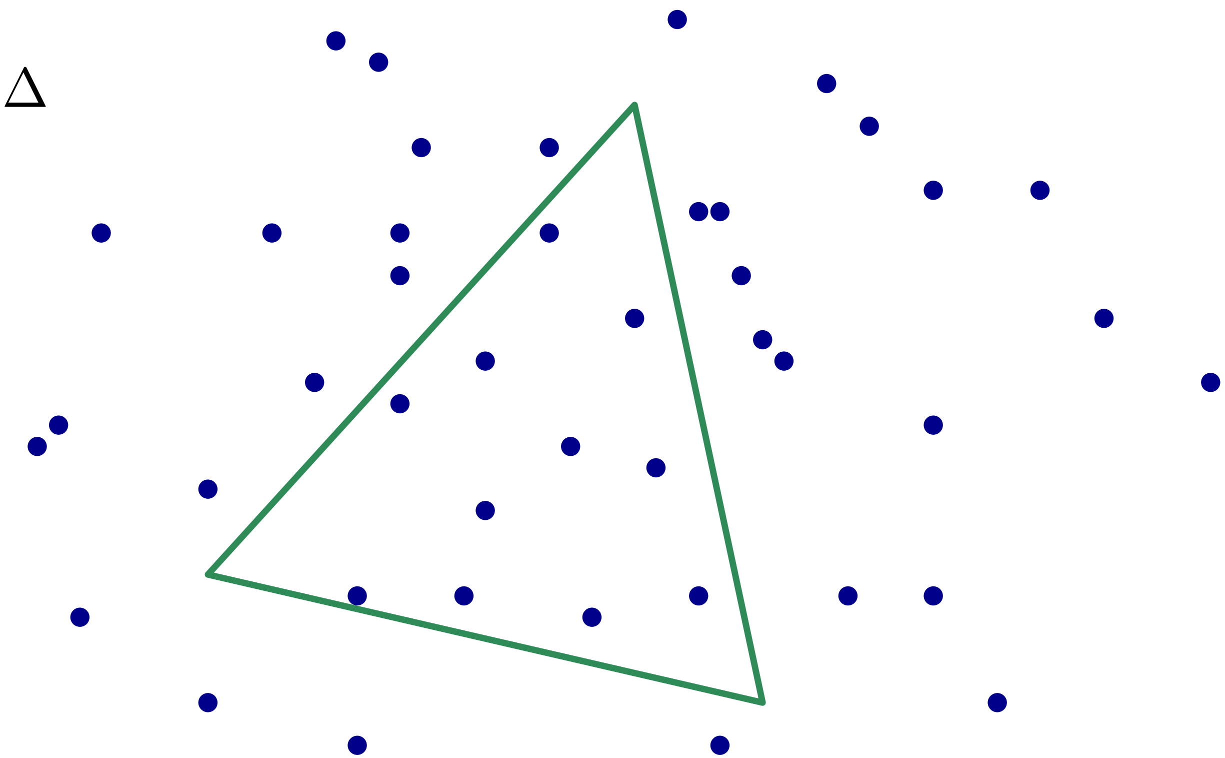

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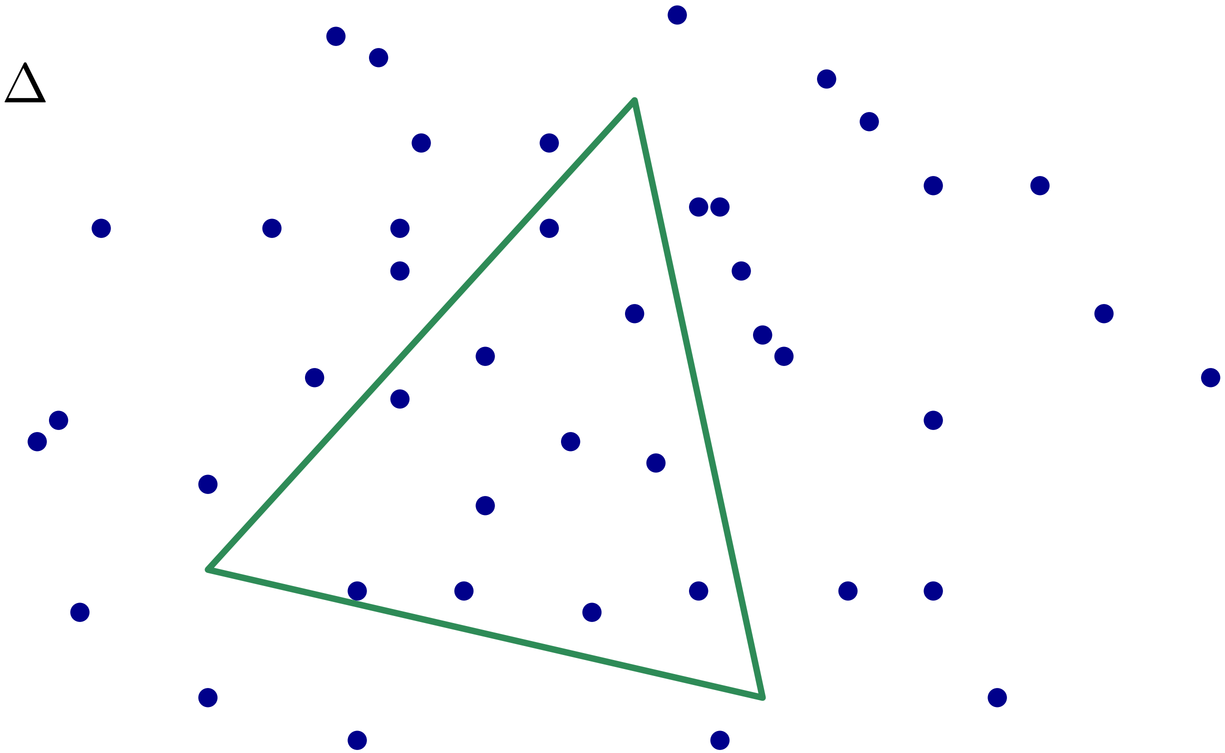

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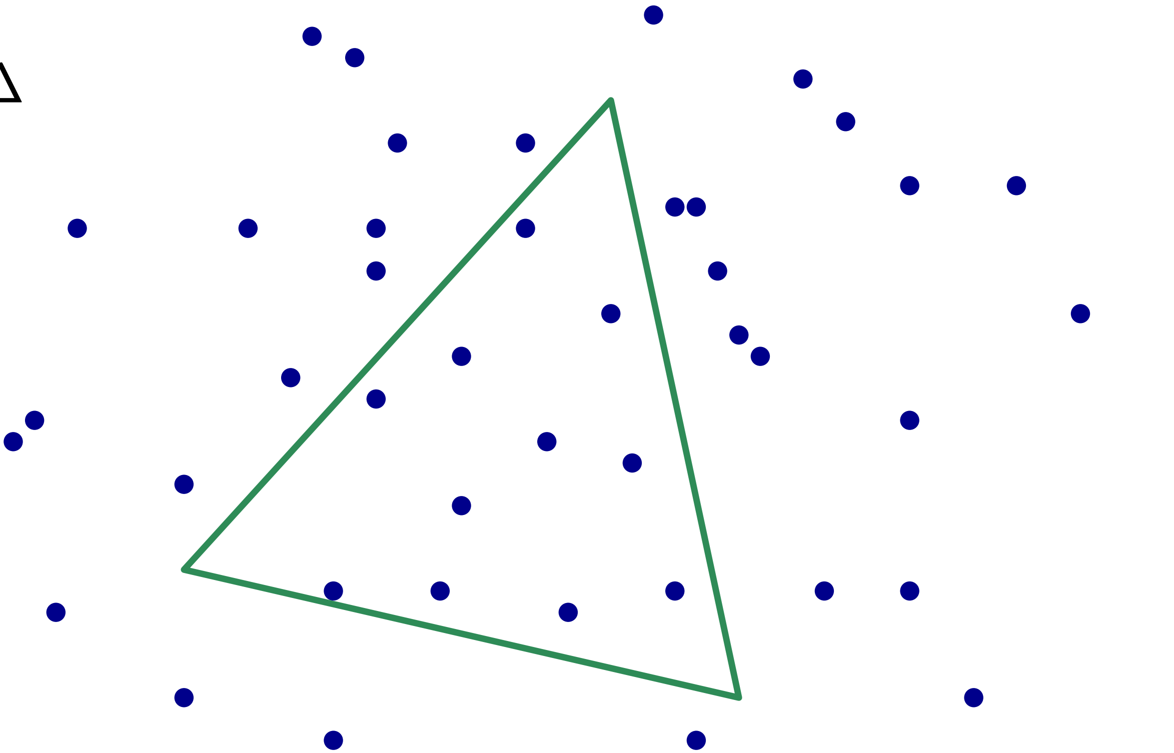

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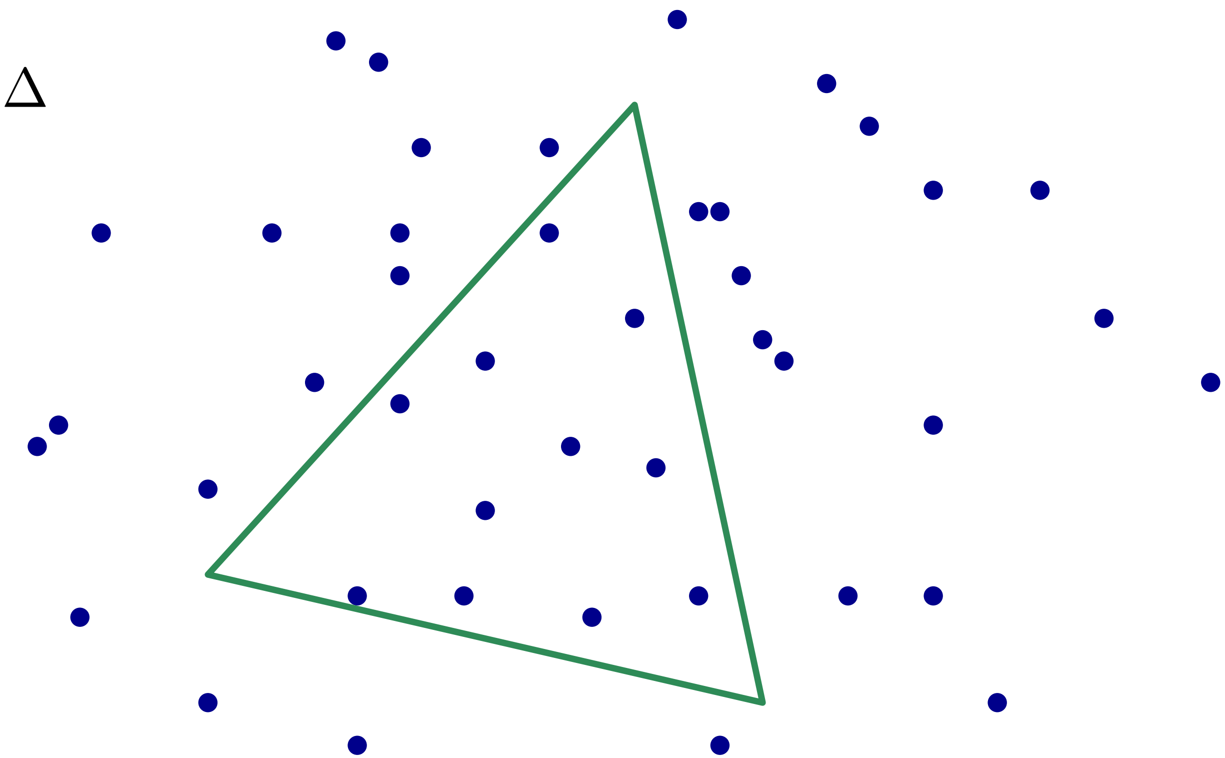

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How do we prove it?

Data Structure Lower Bounds

Theorem we want to prove

Assume we have a data structure:

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Theorem we want to prove **Impossibility result!**

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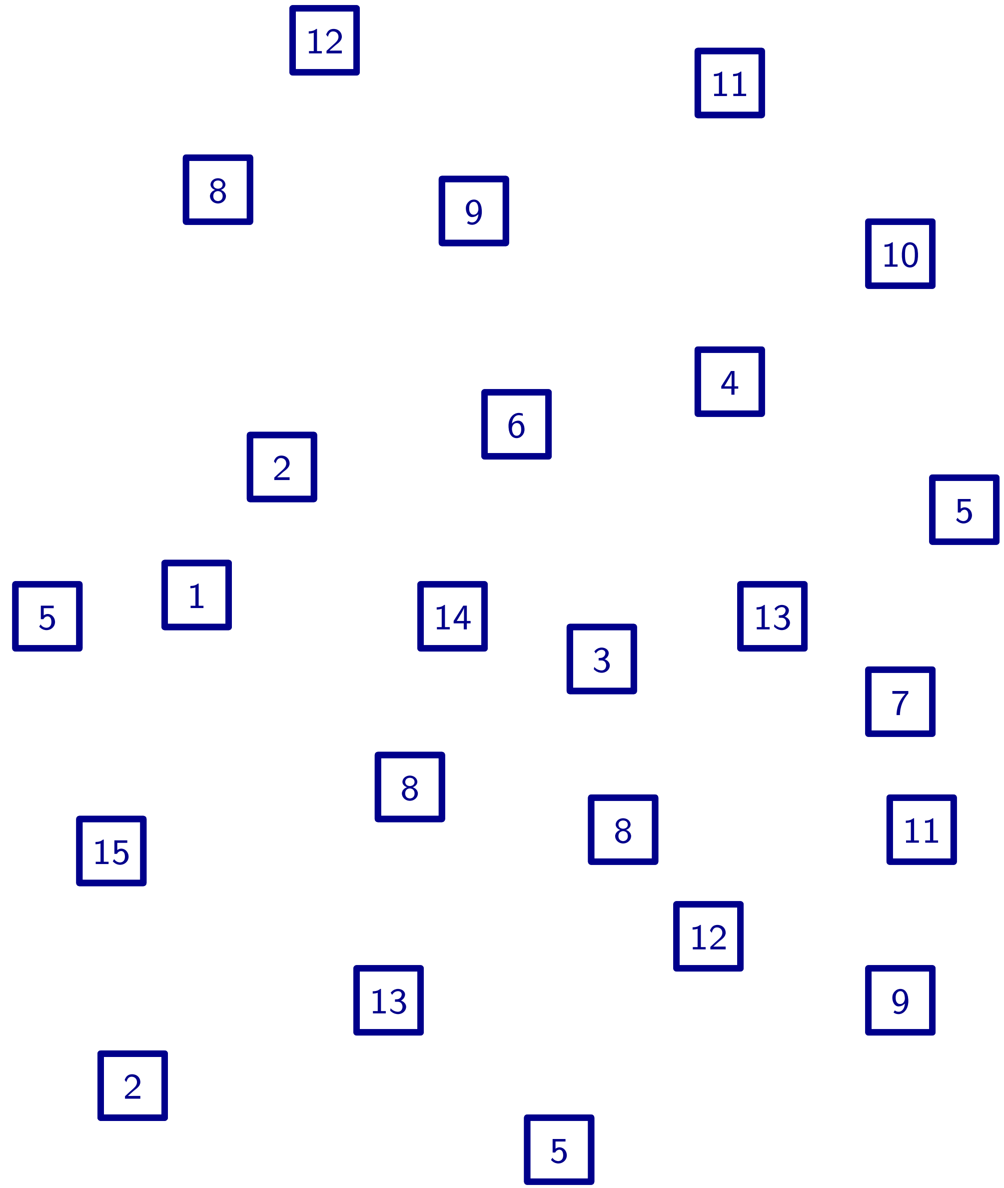
The Model of Computation: A Pointer Machine

Assume, the input is a set P of n items (e.g., points)

DS:

- Storage is a collection of cells
- A cell stores **one** item
- A cell points to two other cells
- There is a special node called the root



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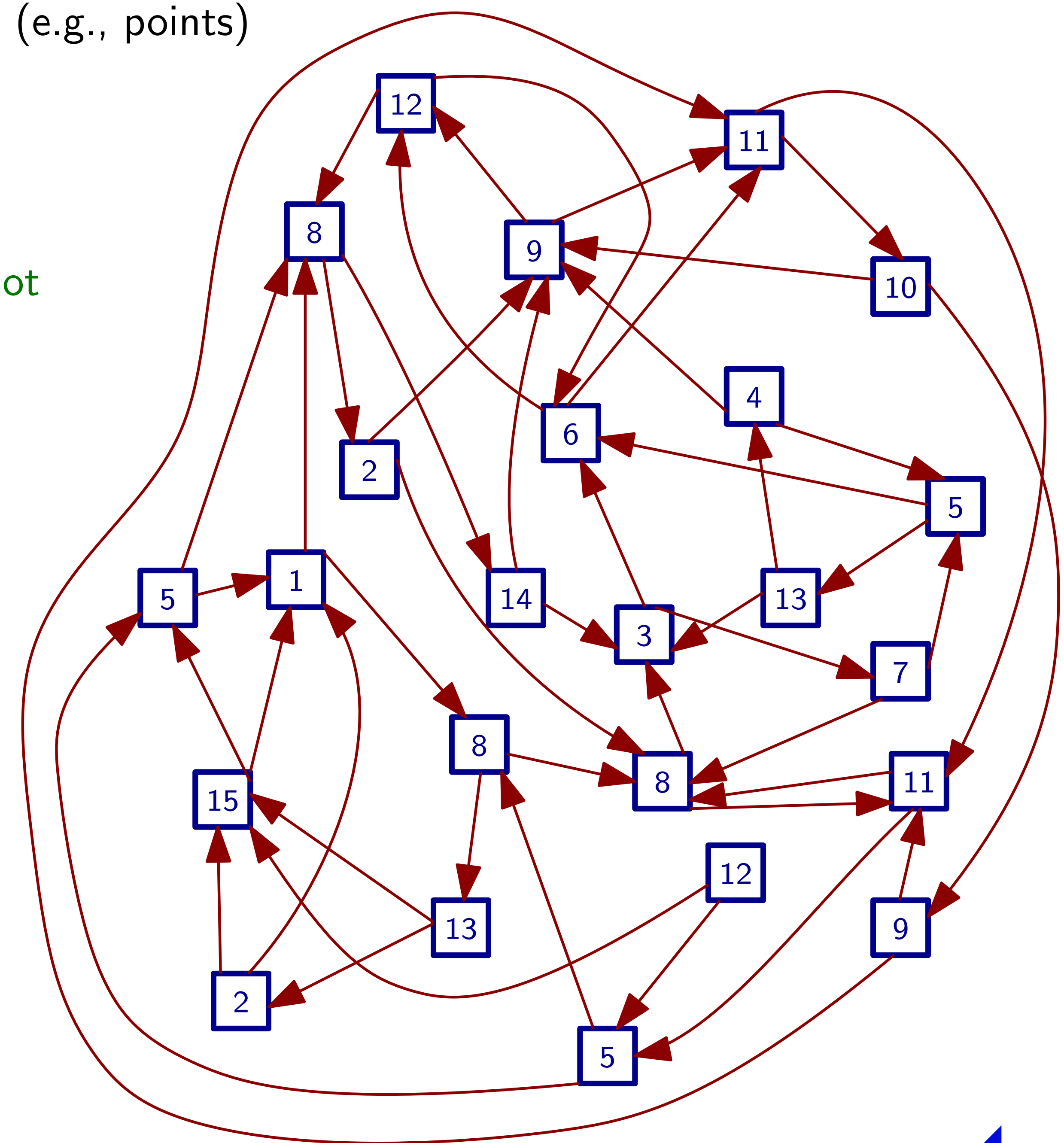

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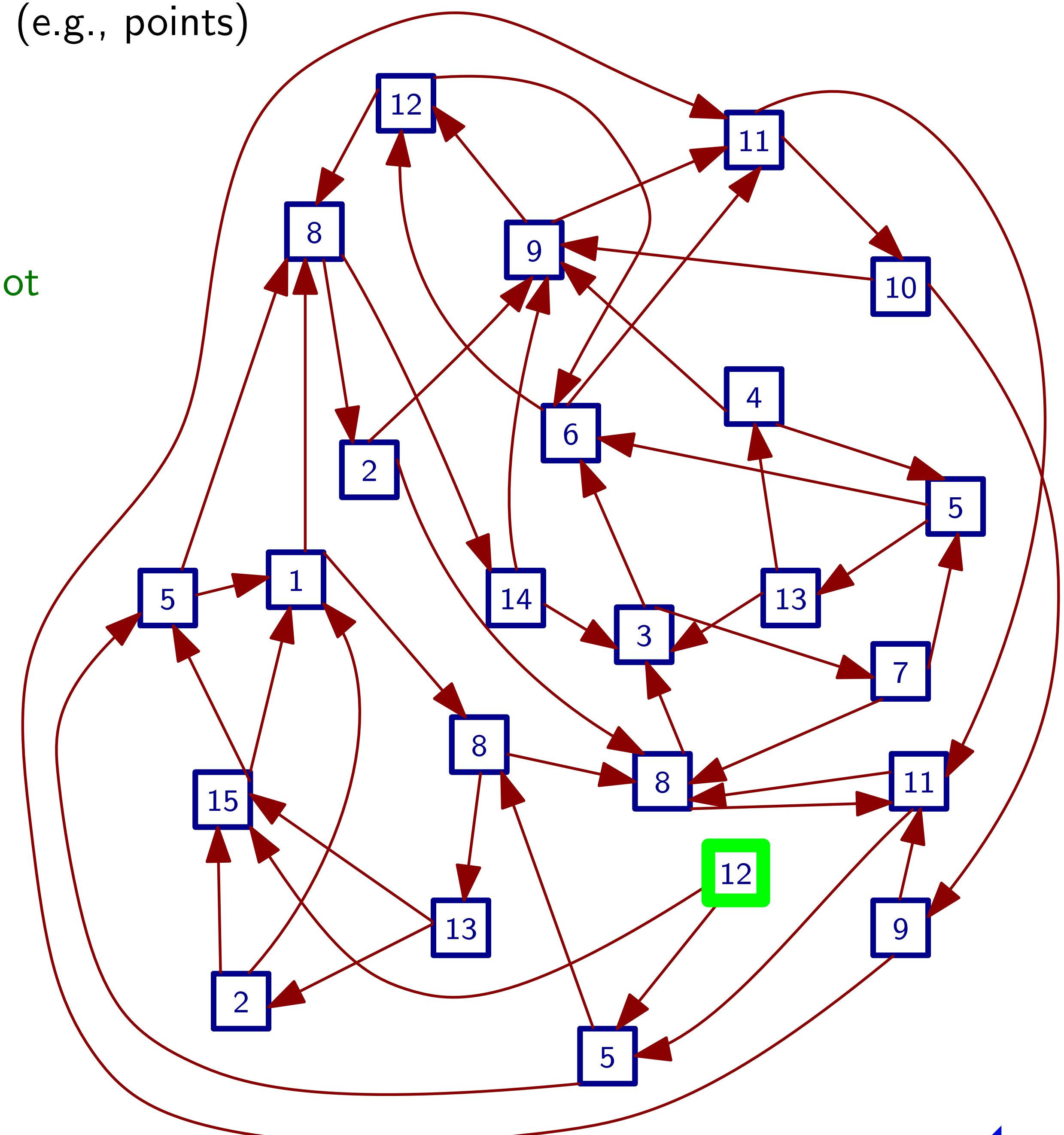

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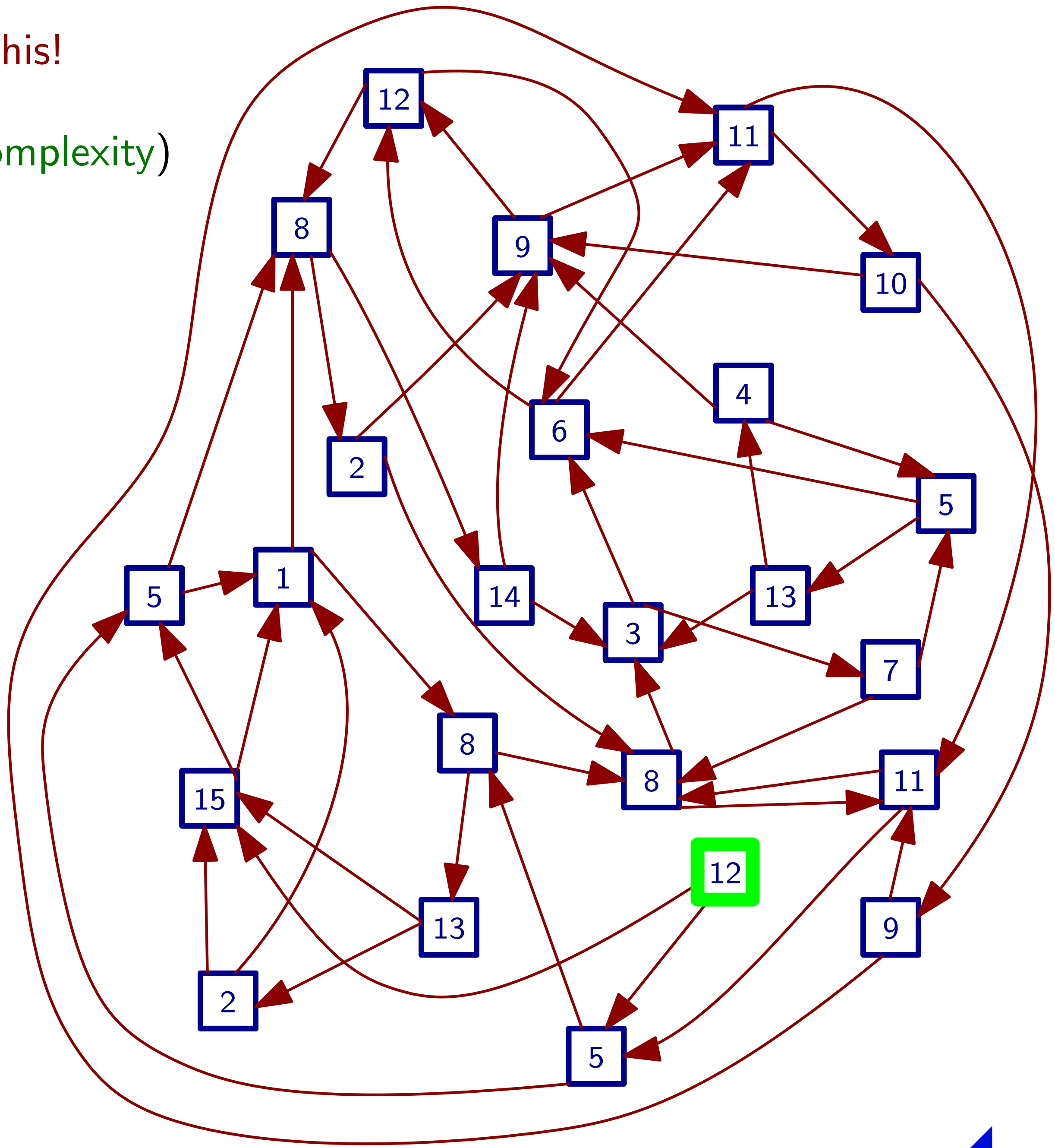

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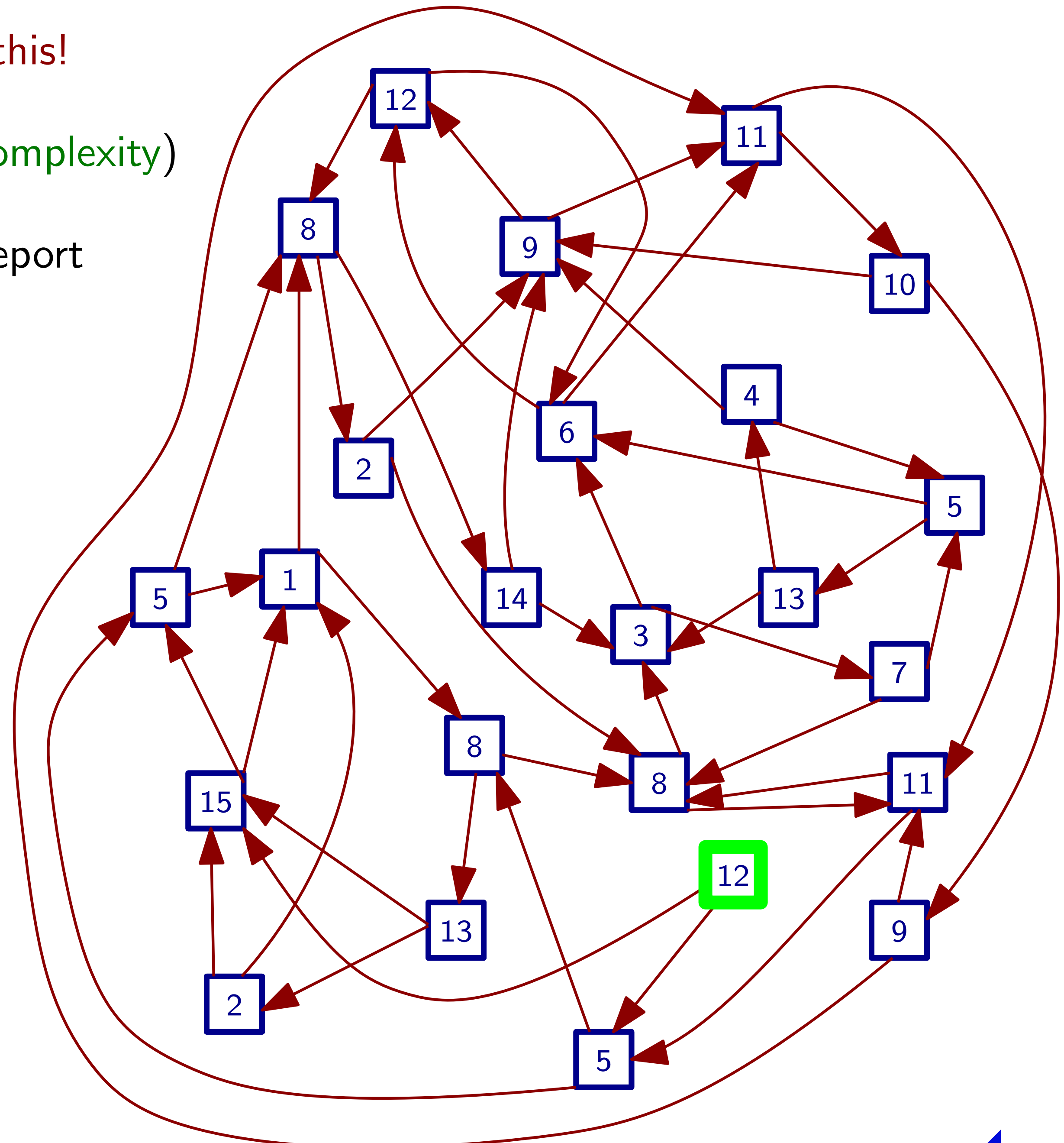

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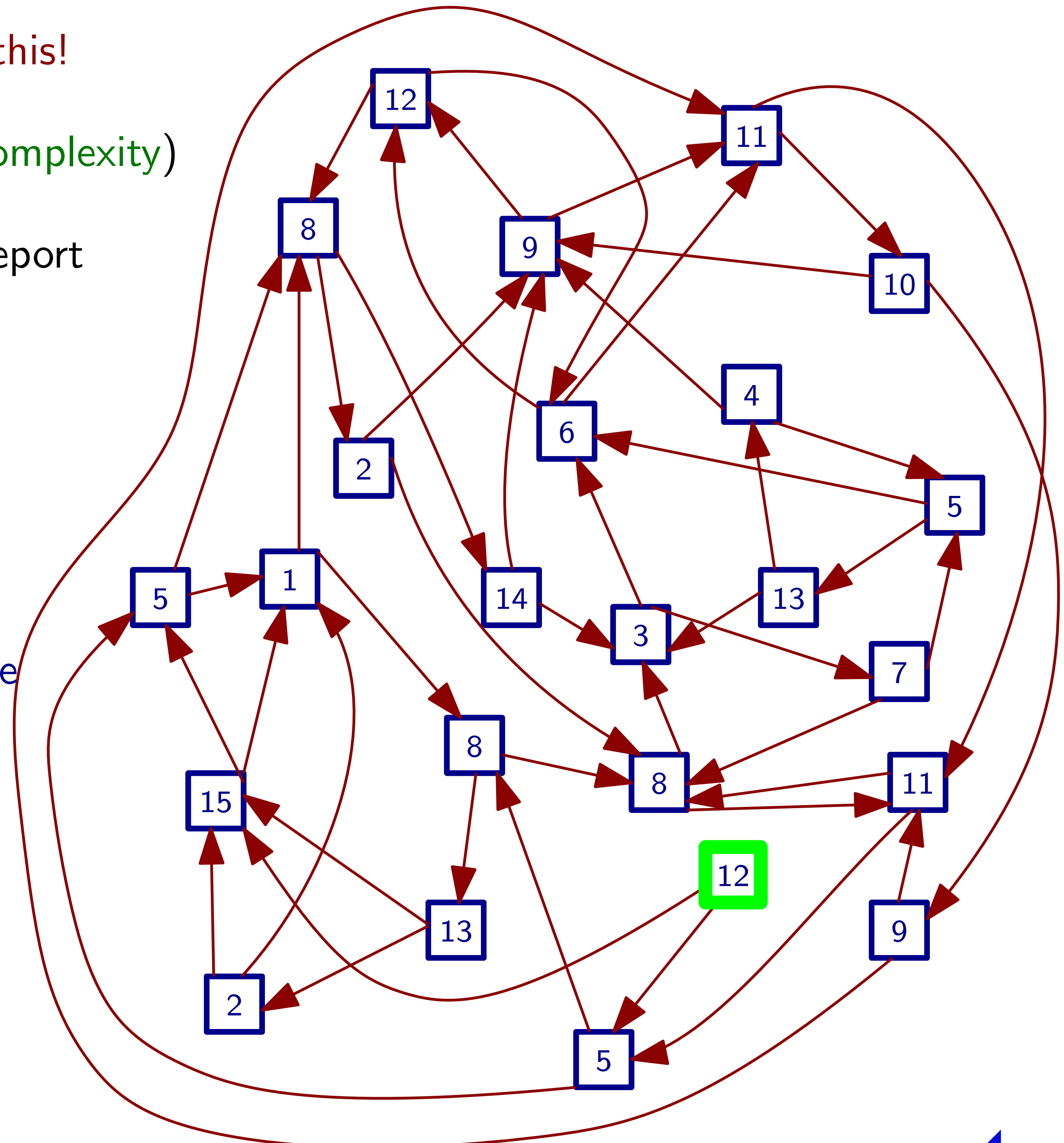

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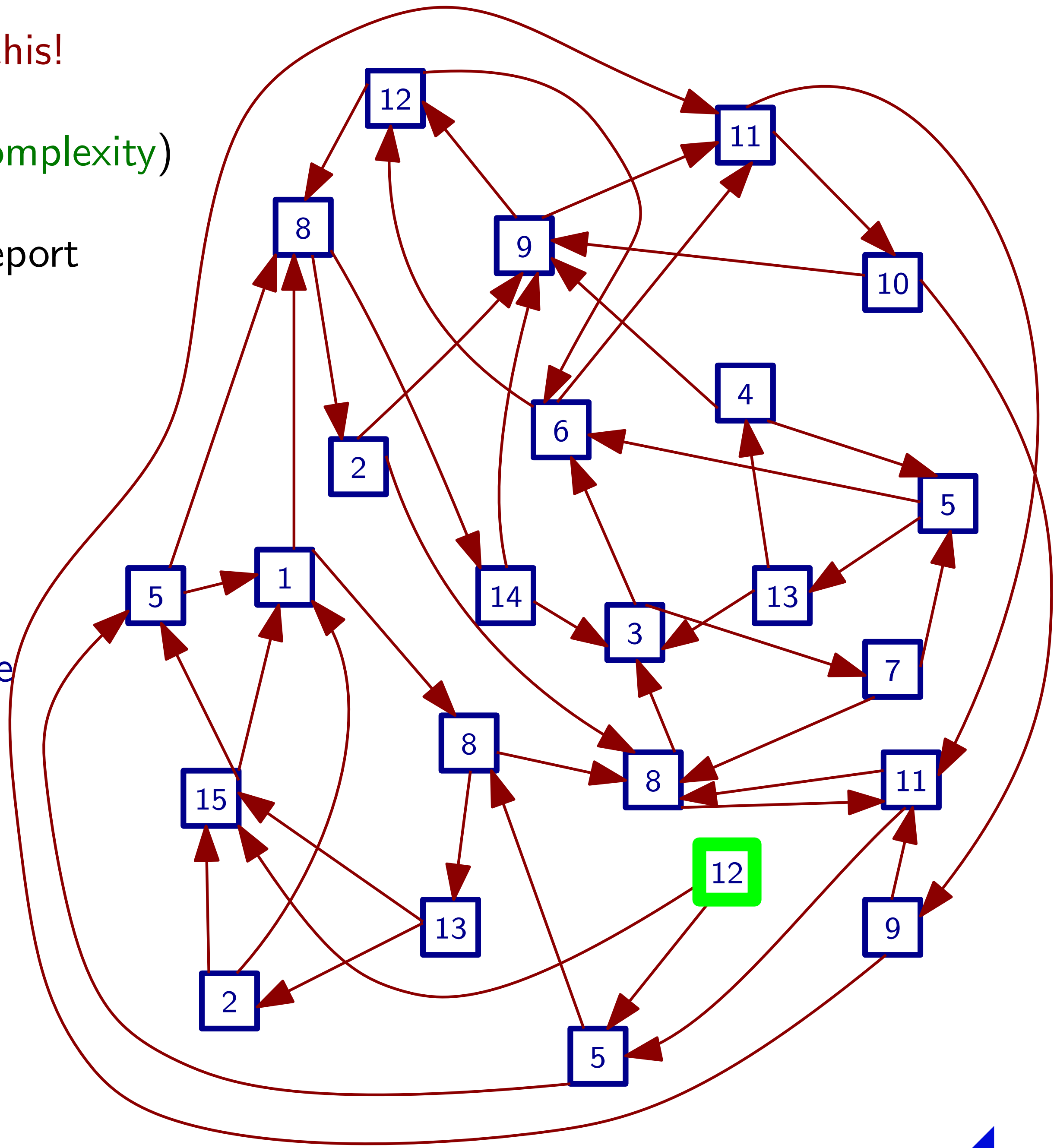
- Computation is free!
- Information is free!



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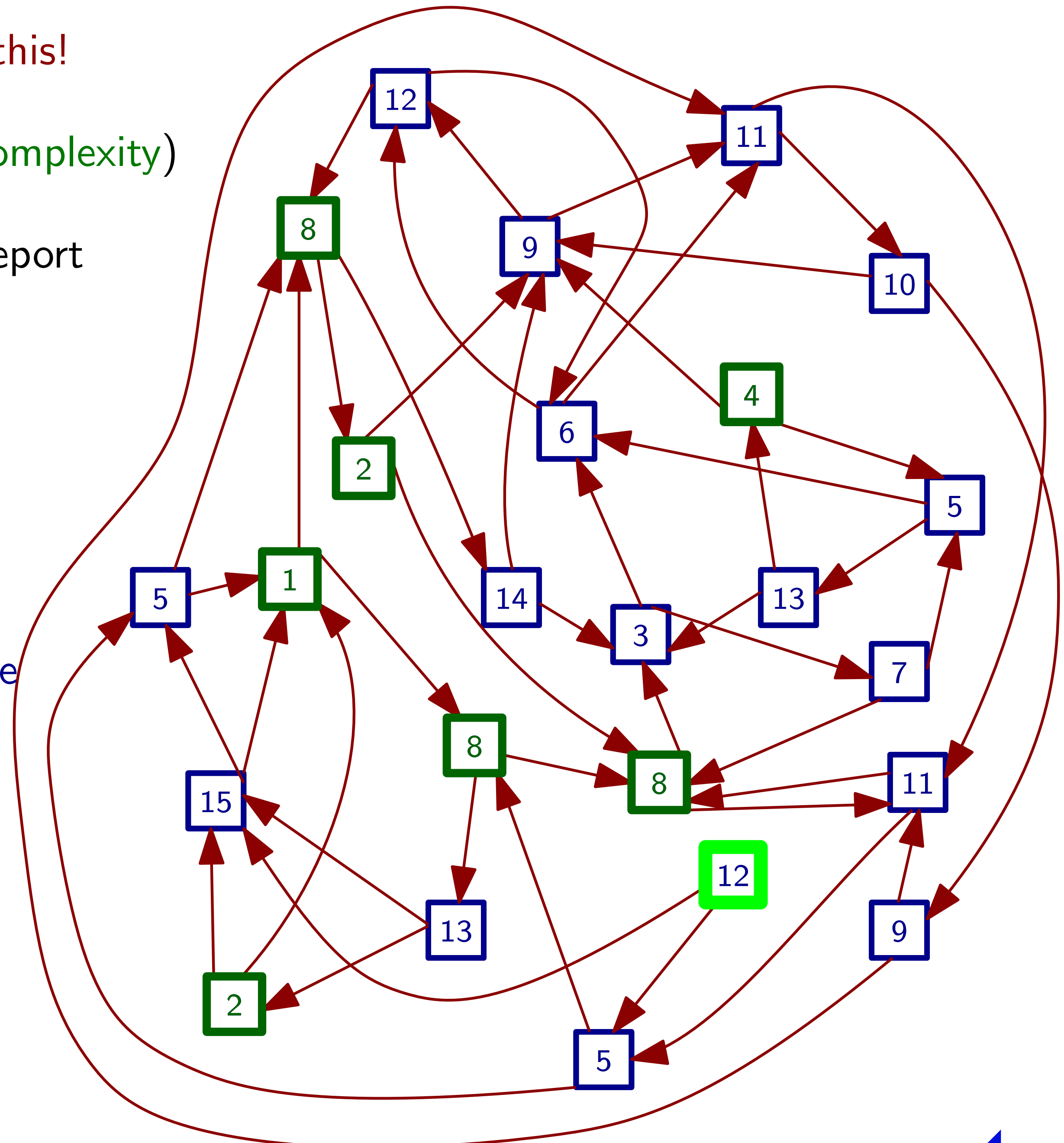

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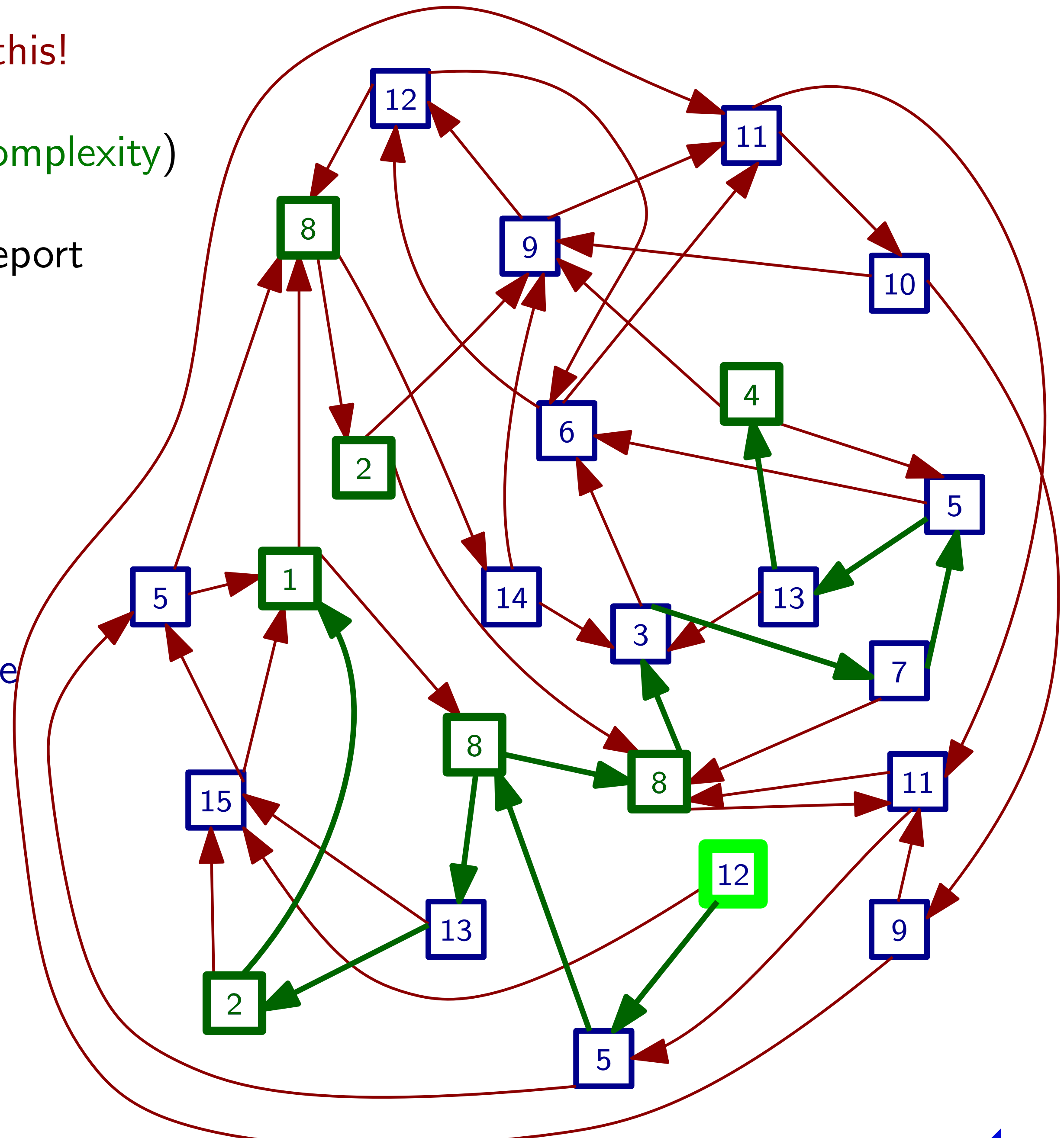

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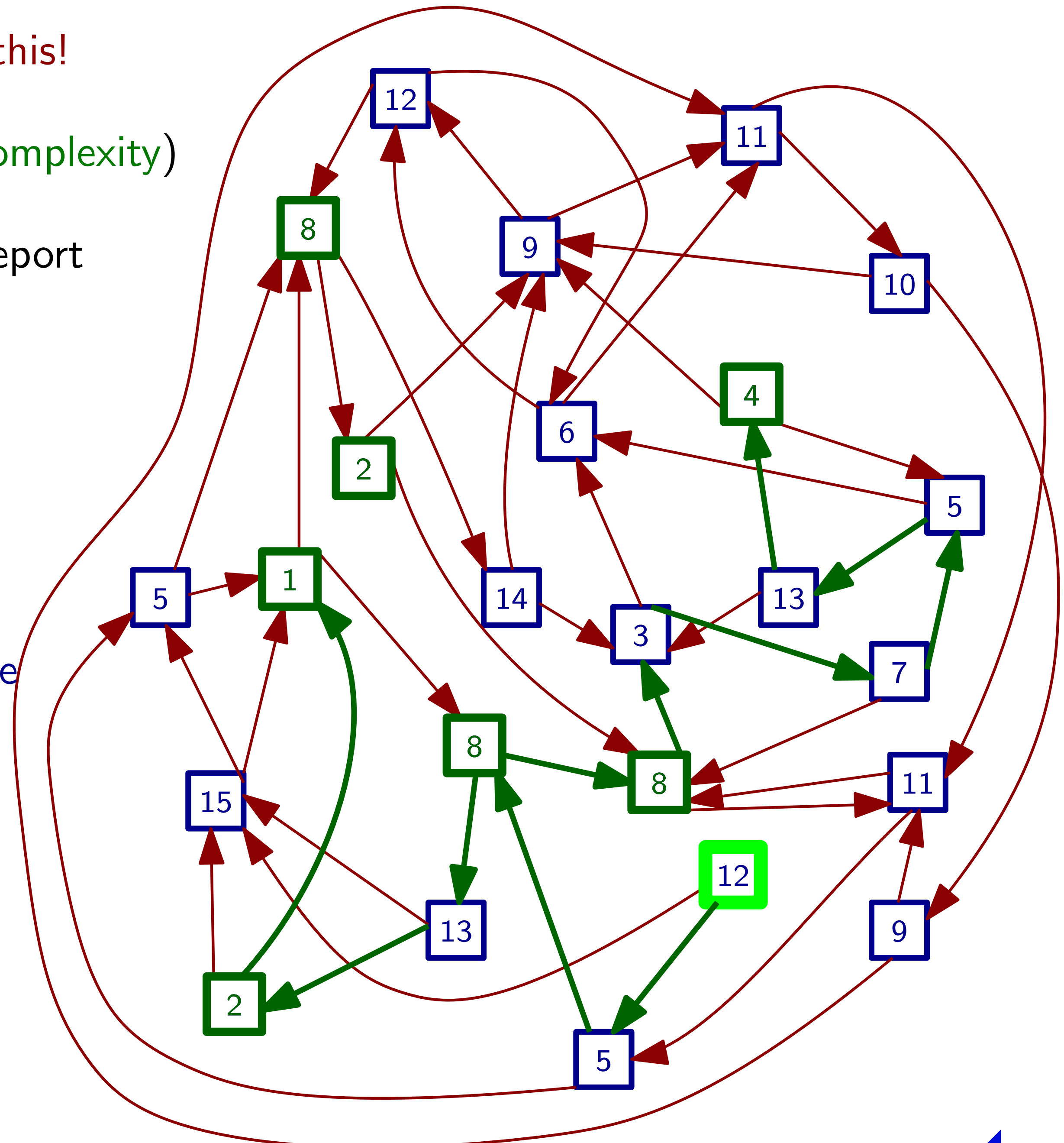

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We used 11 pointers $\Rightarrow$ query time at least 11

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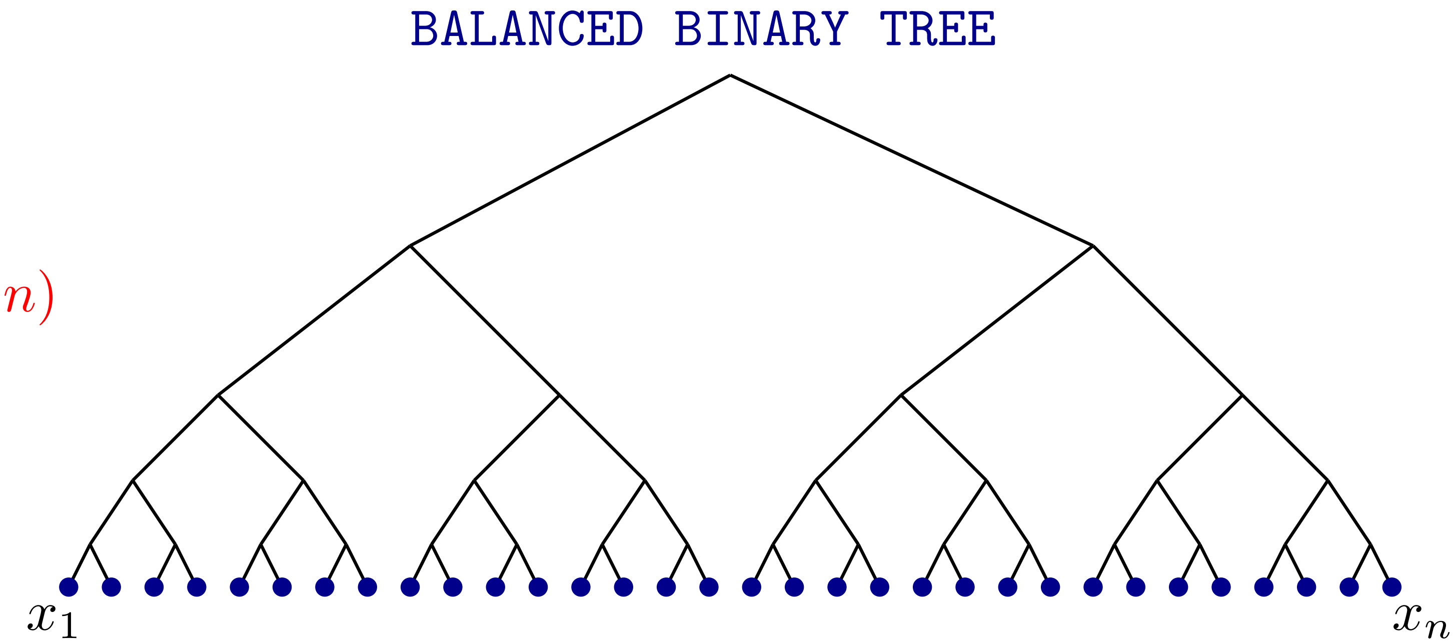
Query: $O(k \log n)$



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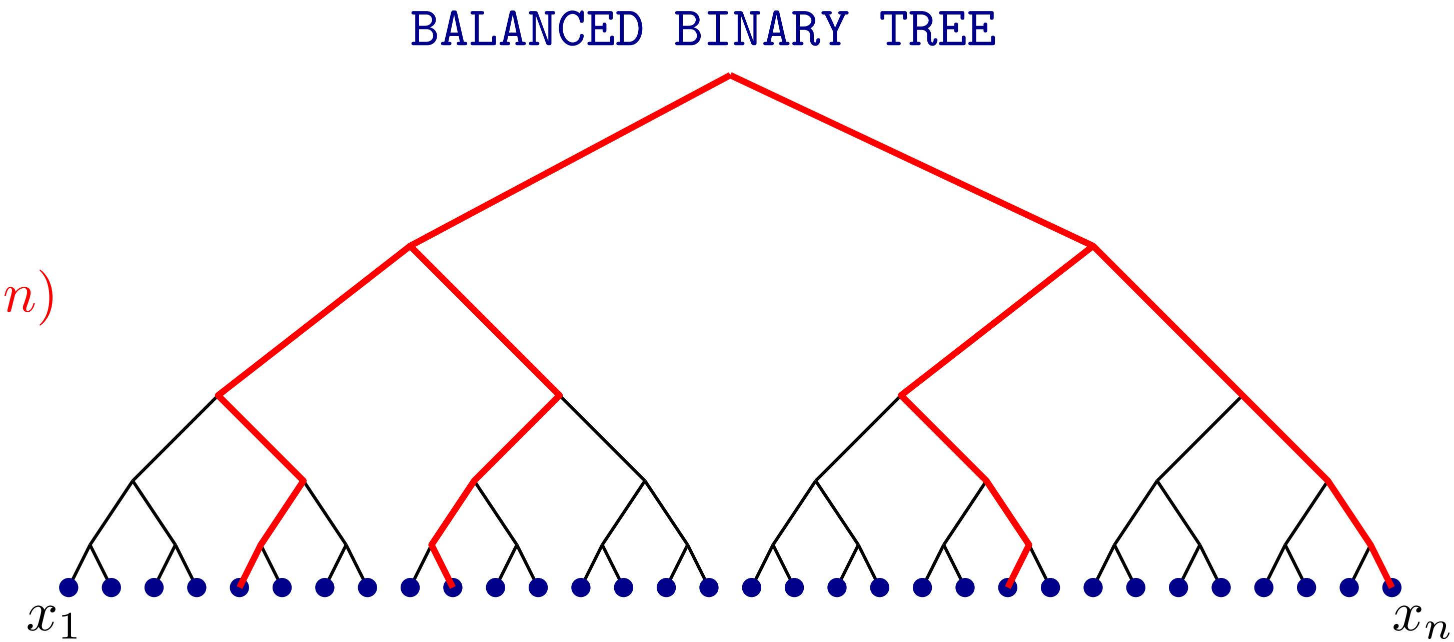
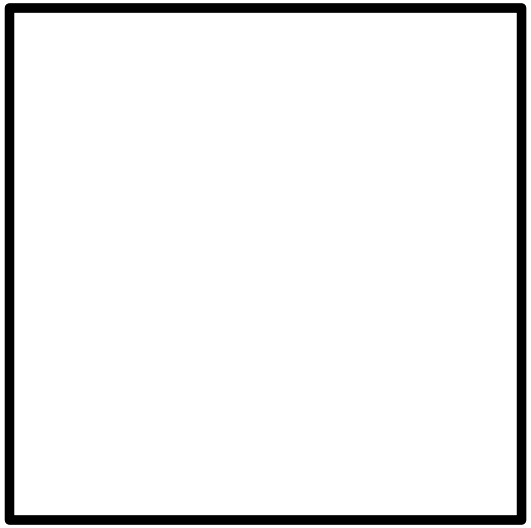


- Query time must be $Q(n) + O(k)$ (or $Q(n) + o(k \log n)$)
- PM can simulate RAM w/ extra $O(\log n)$ factor
  - LB in PM with $Q(n) + O(k \log n) \Rightarrow Q(n)/\log n + O(k)$ LB in RAM

A Framework Theorem

Unit square in 2D



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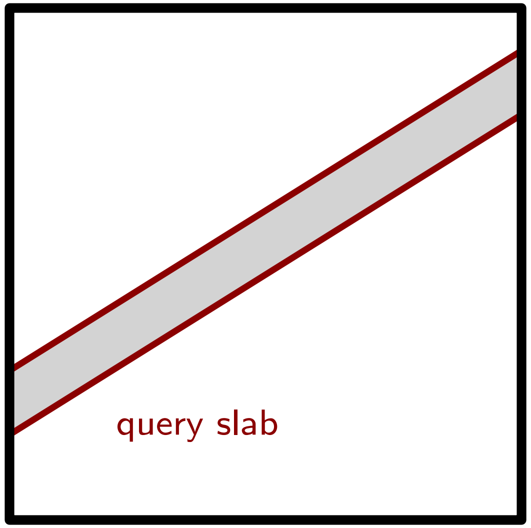

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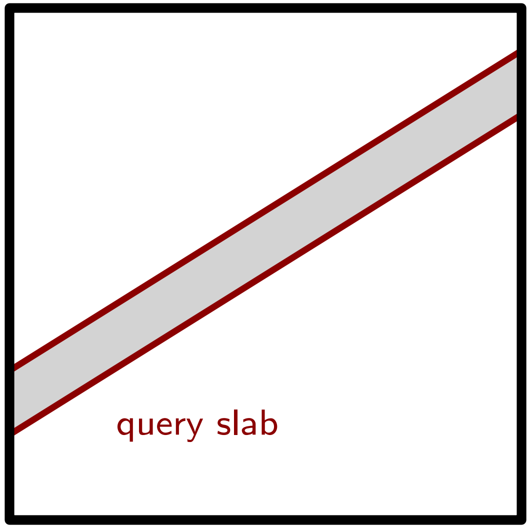

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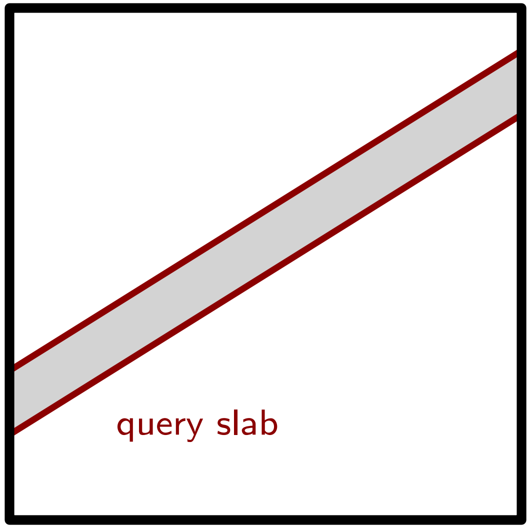

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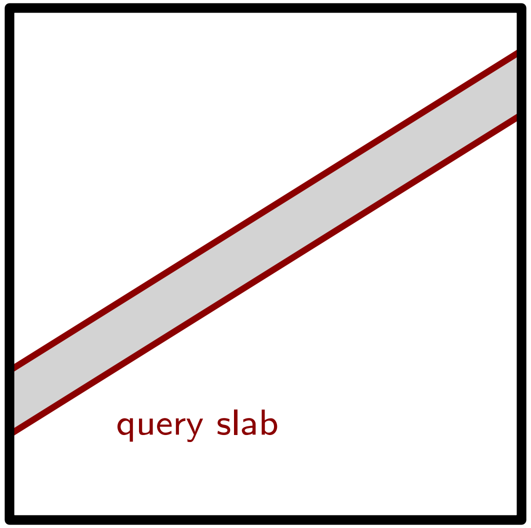

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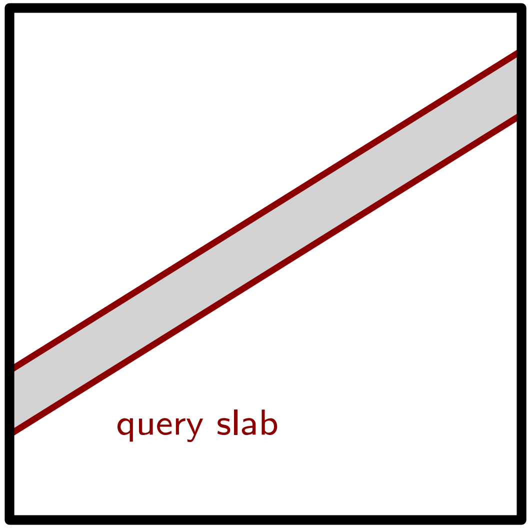

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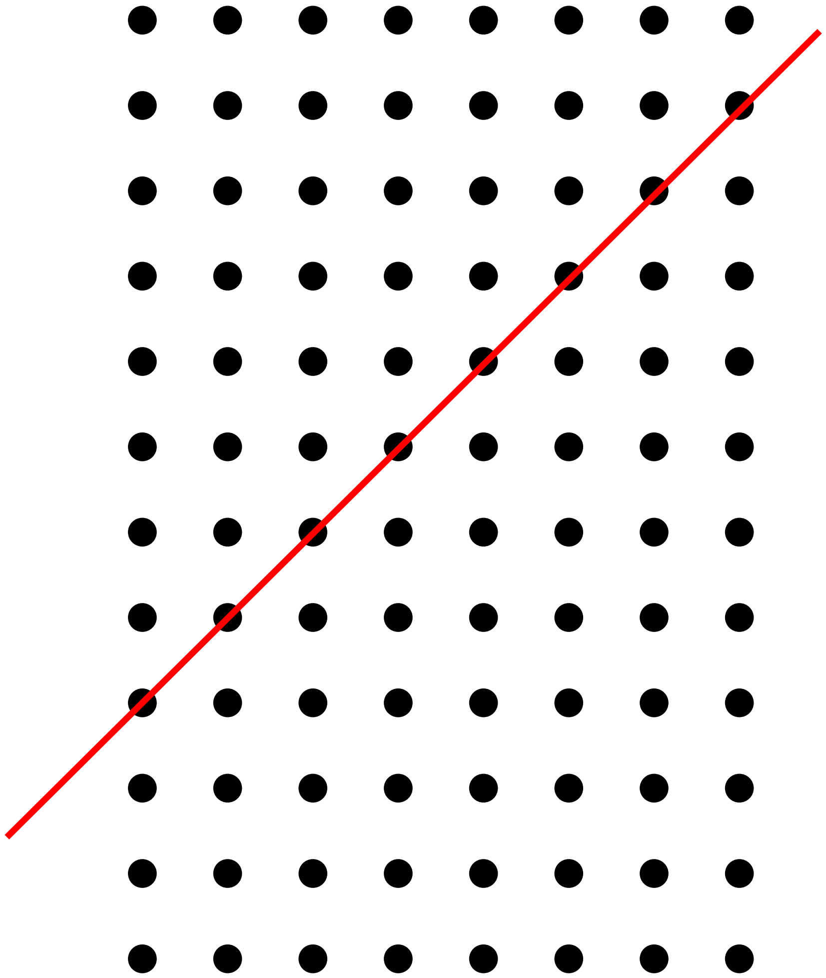
A Discrete Geometry View

- Input: n points
- Query: lines



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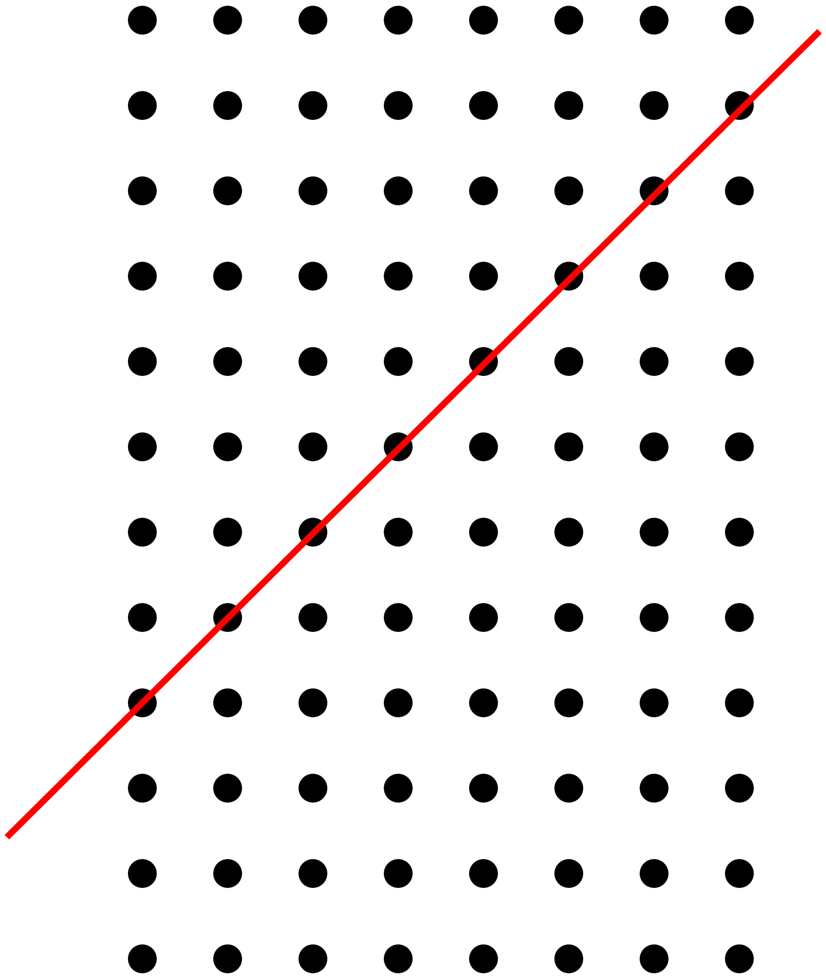

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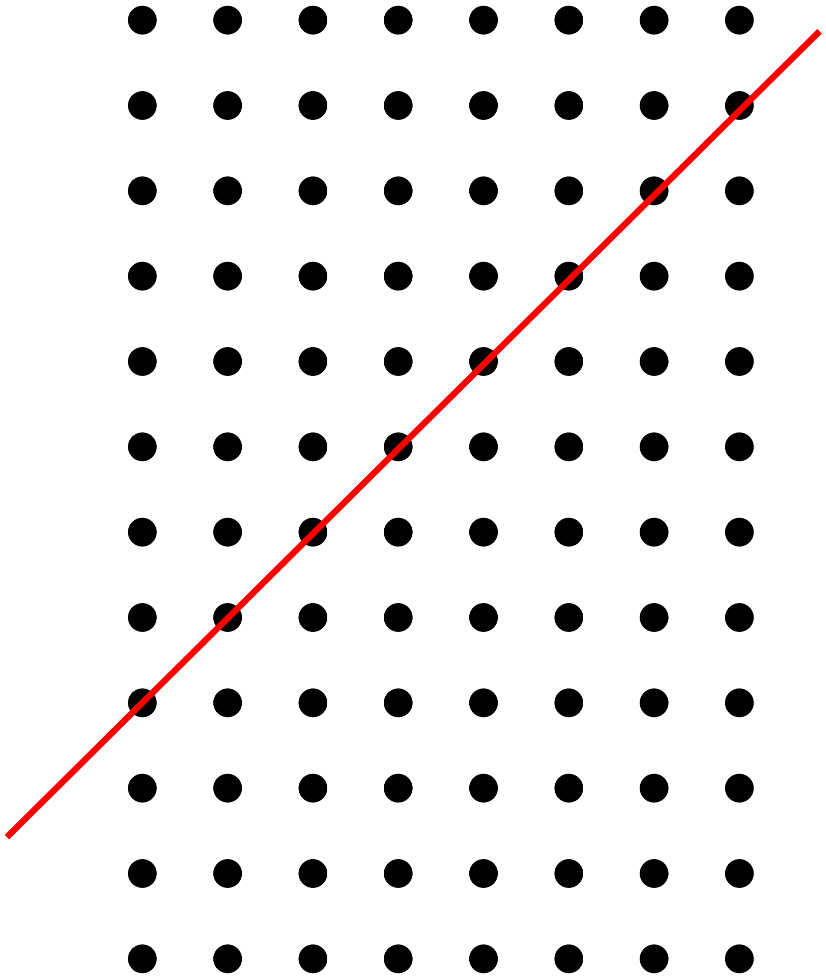
- No $K_{\alpha,\beta}$ in incidence graph
- Lower bound: $S(n) \gg \frac{\text{\# of incidences}}{\alpha 2^{O(\beta)}}$



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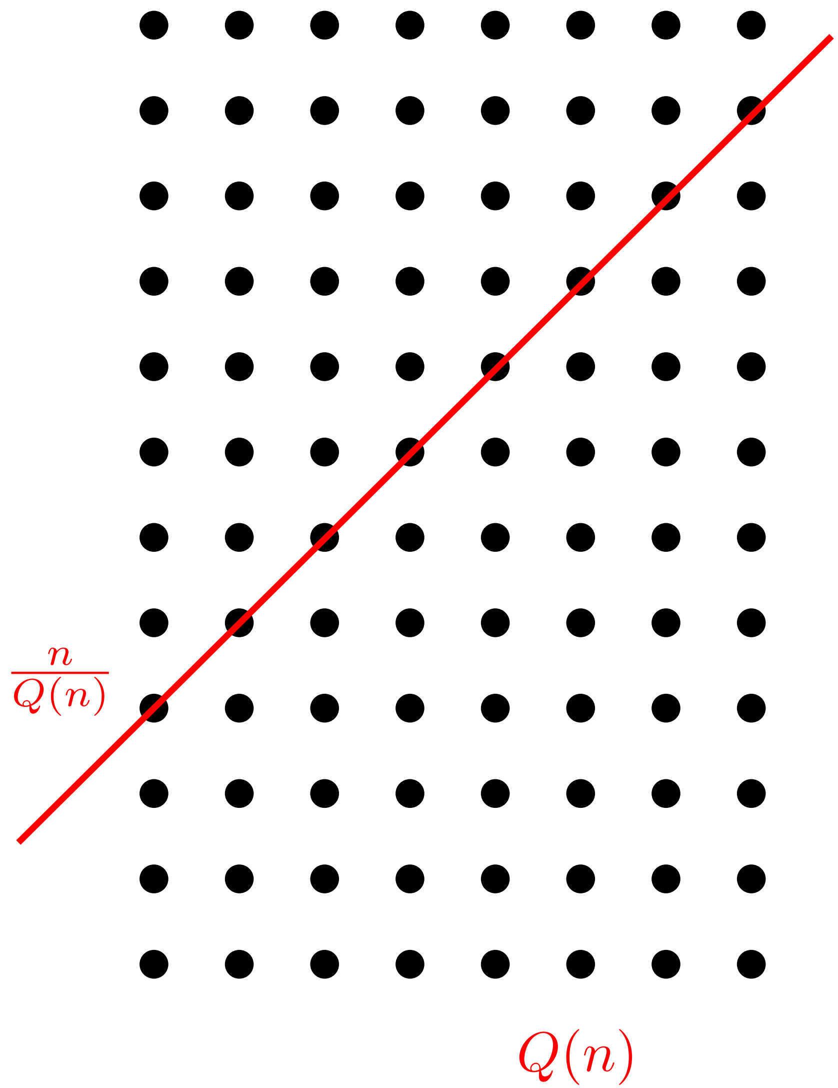
Well-known construction:



A Discrete Geometry View

- Input: n points
- Query: lines

Build:

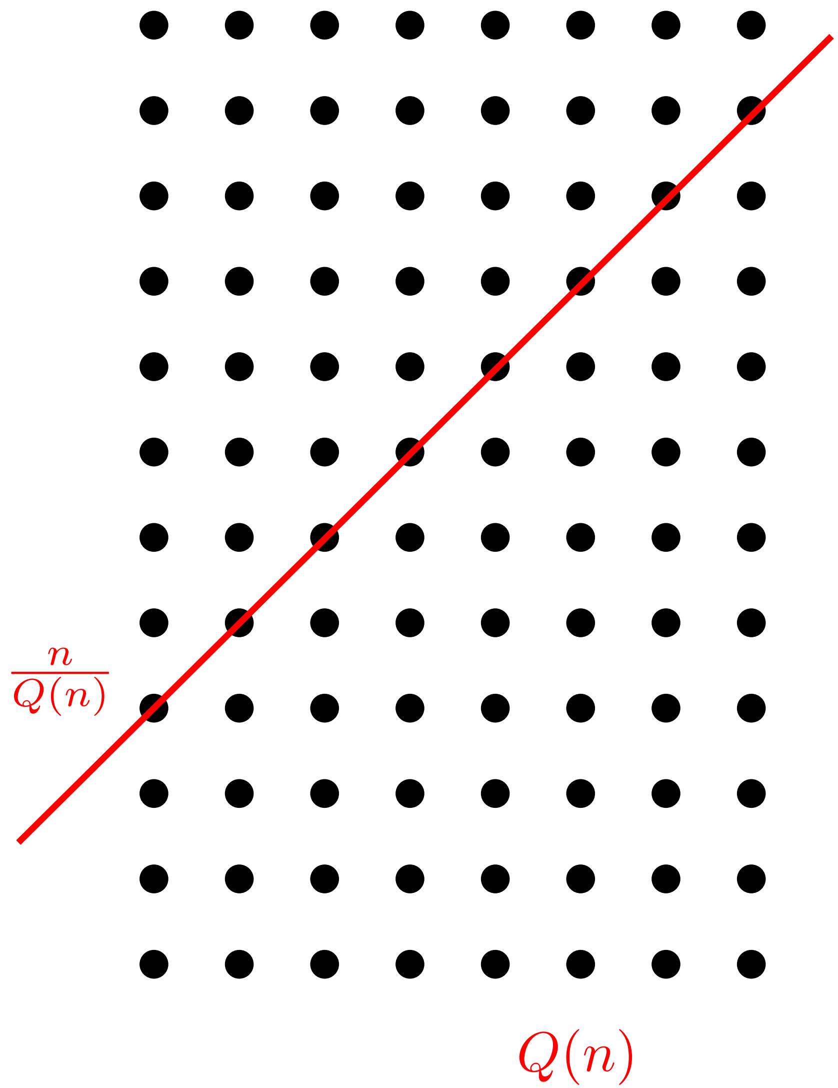
- n points
- (a lot of) m query regions, $r_1, \ldots, r_m$
- (Cond. I) Every r_i contains $\Omega(Q(n))$ points
- (Cond. II) Any α queries contain at most β points.

$$S(n) = \Omega\left(\frac{\sum |r_i|}{\alpha 2^{O(\beta)}}\right)$$

- Every line is $Q(n)$-rich
- No $K_{\alpha,\beta}$ in incidence graph
- Lower bound: $S(n) \gg \frac{\#\text{ of incidences}}{\alpha 2^{O(\beta)}}$

Well-known construction:

Slopes of 1,2,3,. . . , $\frac{n}{Q^2(n)}$



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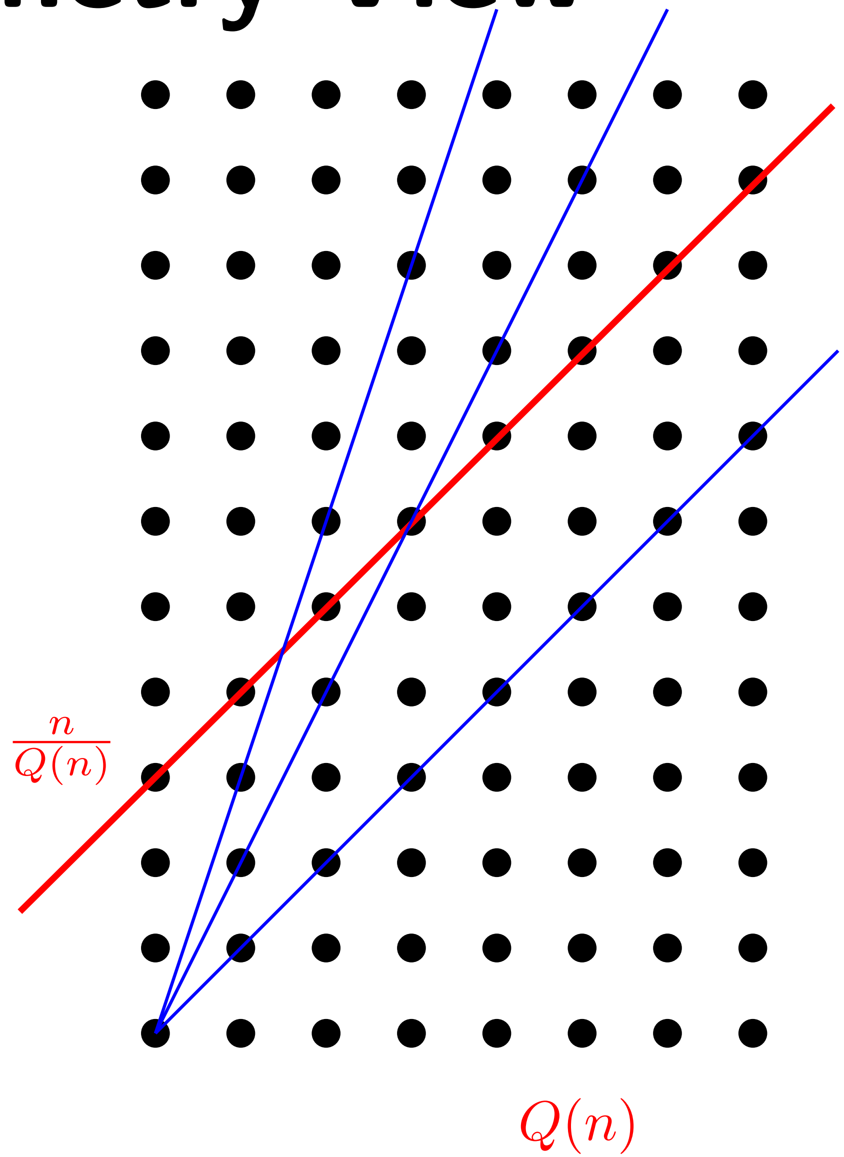
$\Omega\left(\frac{n}{Q(n)}\right)$ values for Y-intersepts



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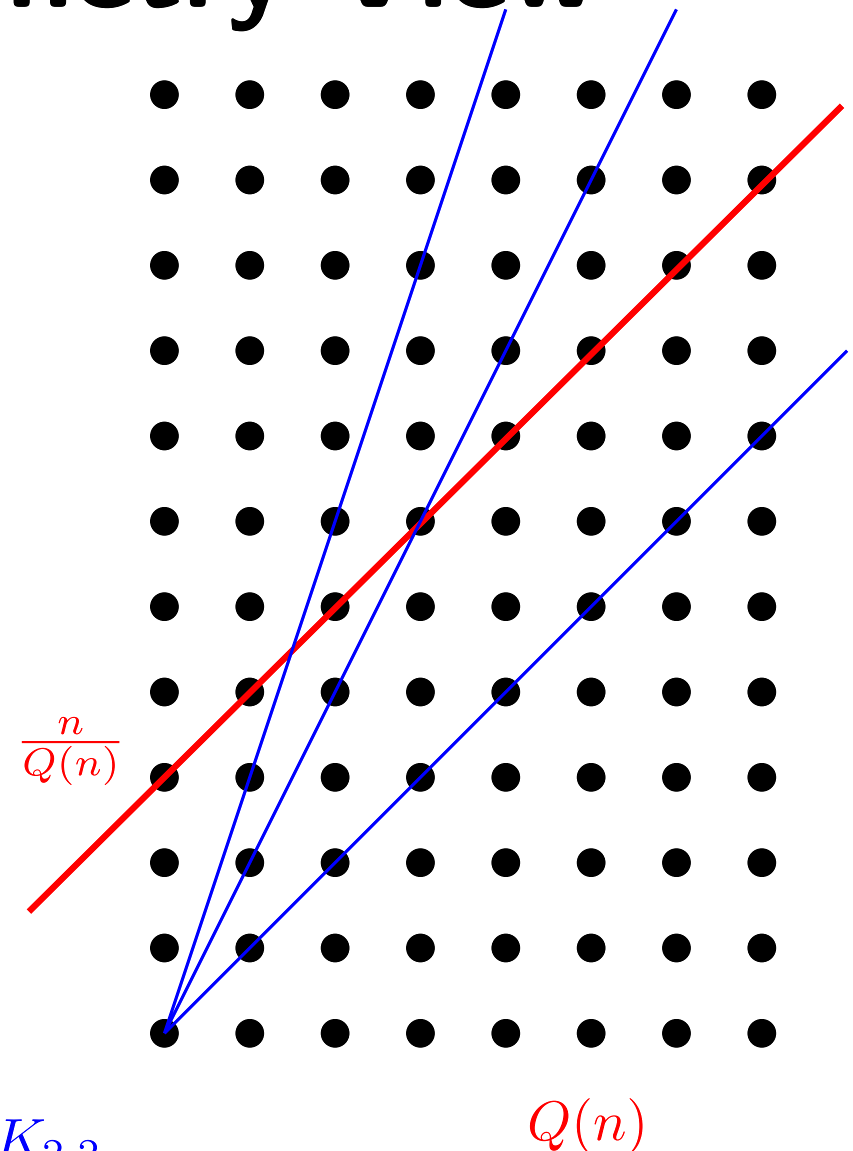

Well-known construction:

Slopes of 1,2,3,$\ldots$, $\frac{n}{Q^2(n)}$

$\Omega\left(\frac{n}{Q(n)}\right)$ values for Y-intersepts

No $K_{2,2}$

$I = \frac{n^2}{Q^2(n)}$ space lower bound

Optimal

A Discrete Geometry View

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- Query: lines

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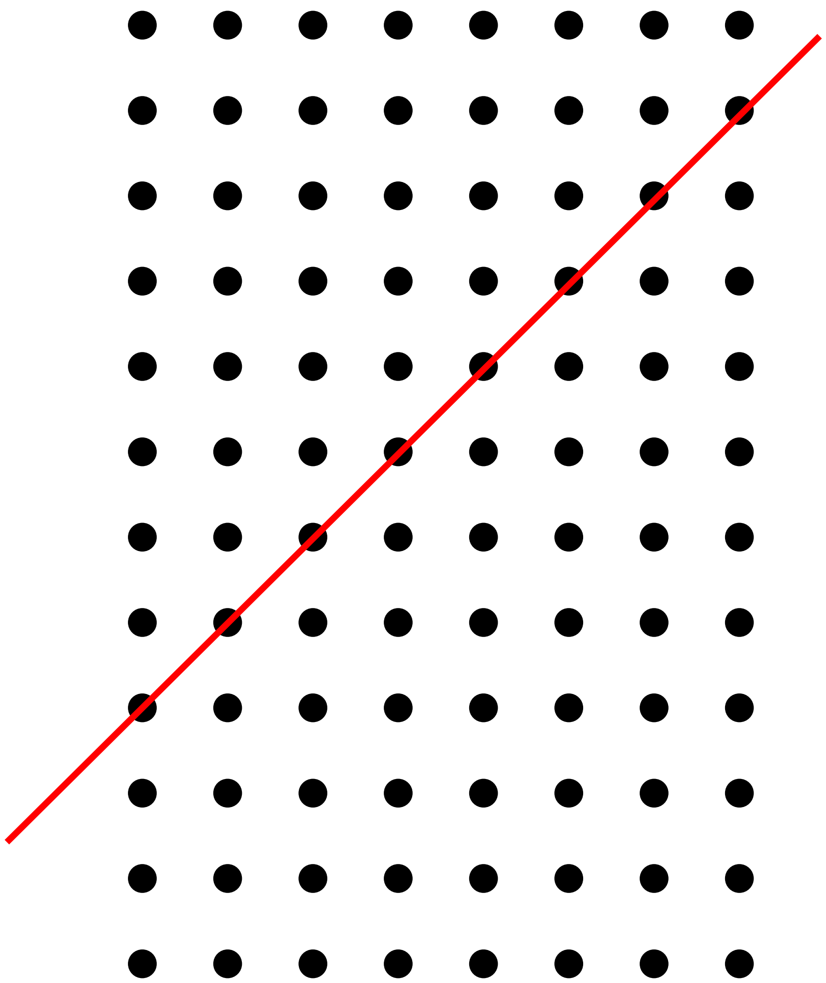

Afshani, Cheng, *SOSA'23*:

$Q(n) \gg \left(\frac{n^2}{S(n)}\right)^{\frac{d-1}{d}}$

For $S(n) = O(n) \Rightarrow Q(n) = \Omega(n^{1-1/d})$

(only **tight** LB for $d > 2$)

Semialgebraic Range Reporting

Input:

- n points in $\mathbb{R}^d$.
- Store in a DS
- Given a range R
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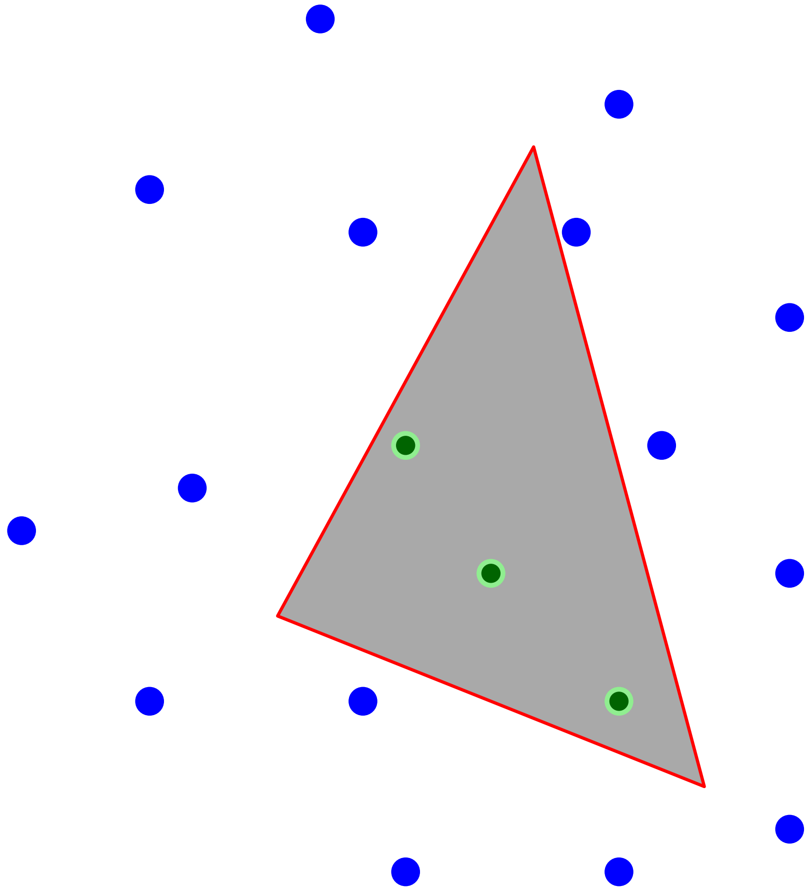

n space, $n^{1-1/d}$ query time (low space)

n^d space, $\log^{d-1} n$ query time (fast query)

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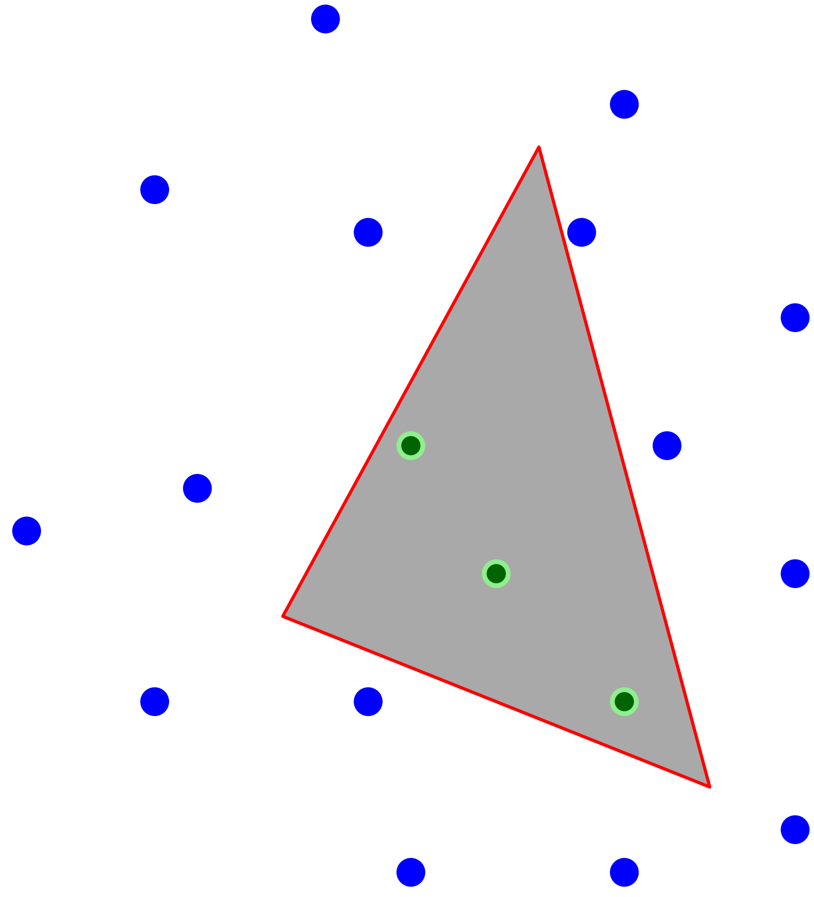

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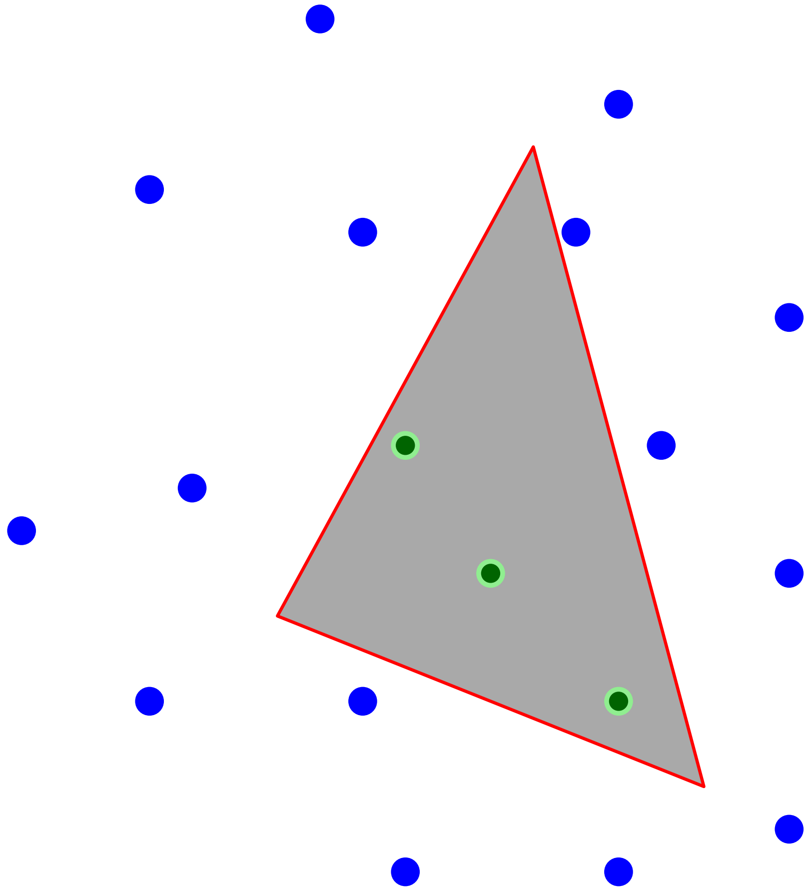

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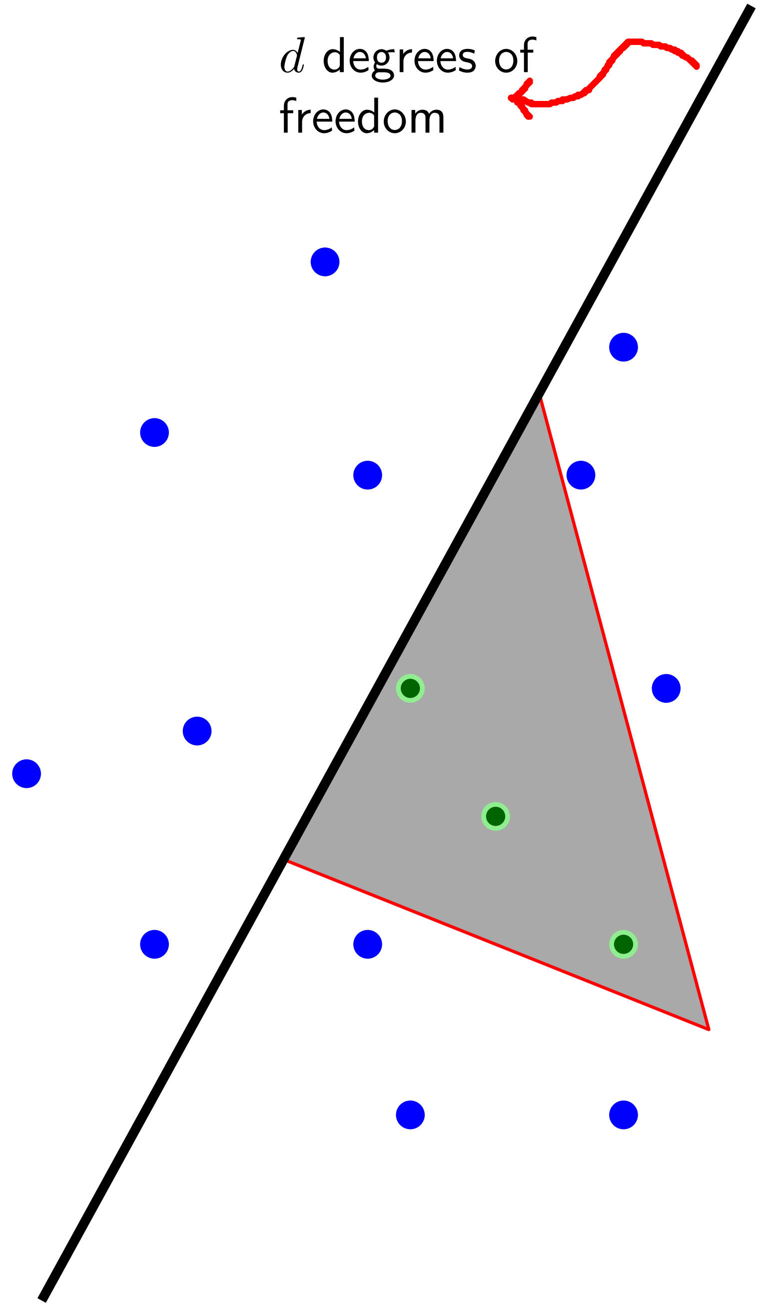

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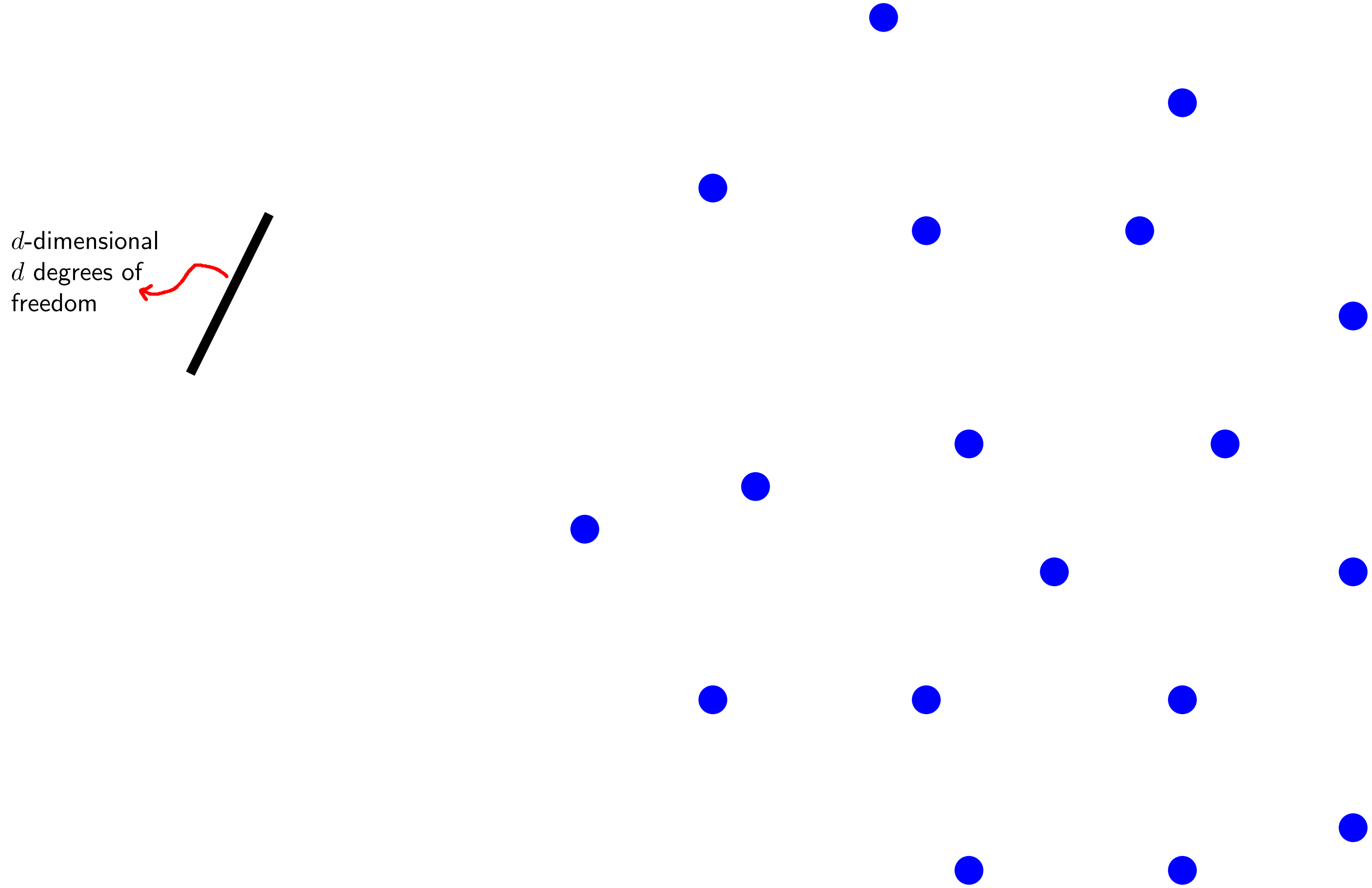

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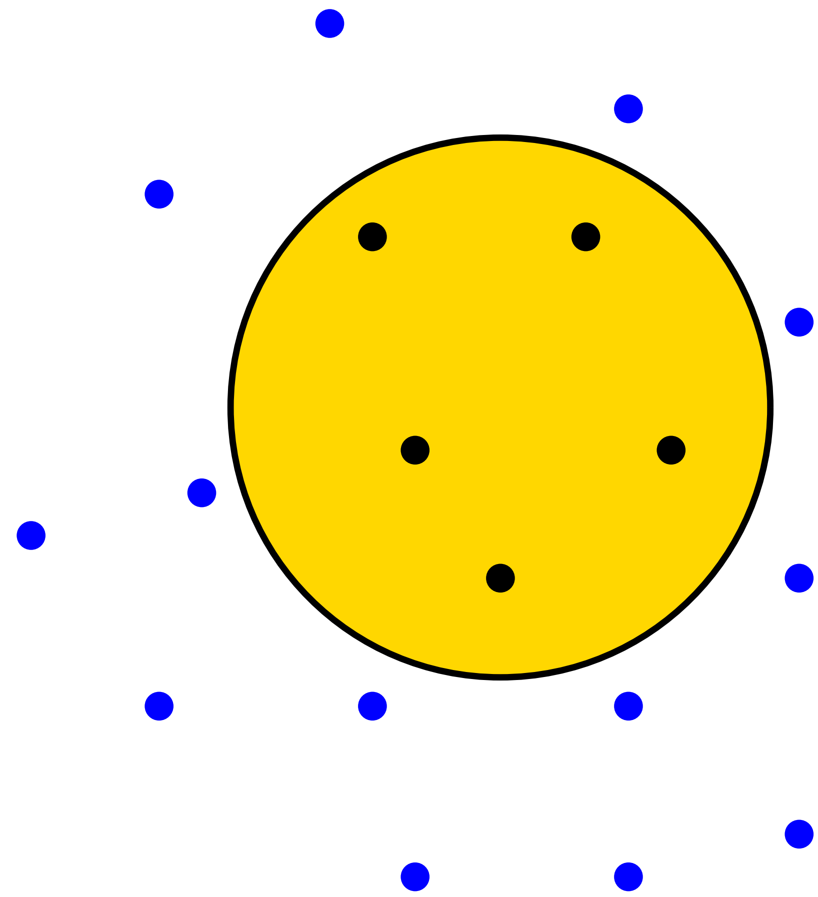
d-dimensional
d degrees of freedom



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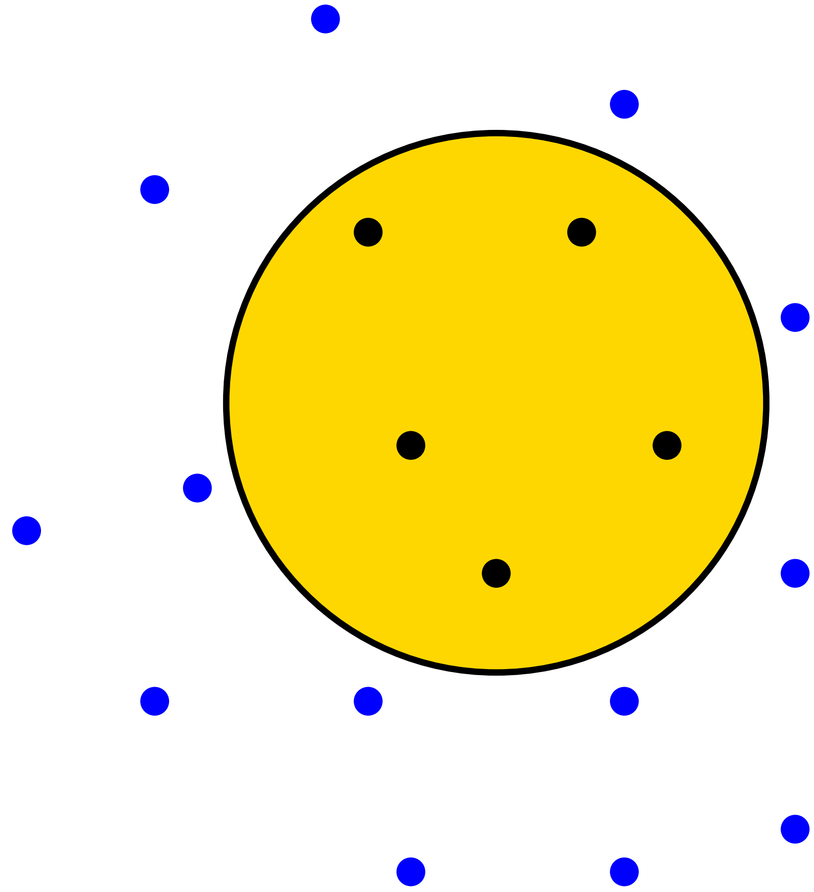
$(x_i - a)^2 + (y_i - b)^2 \leq r^2$



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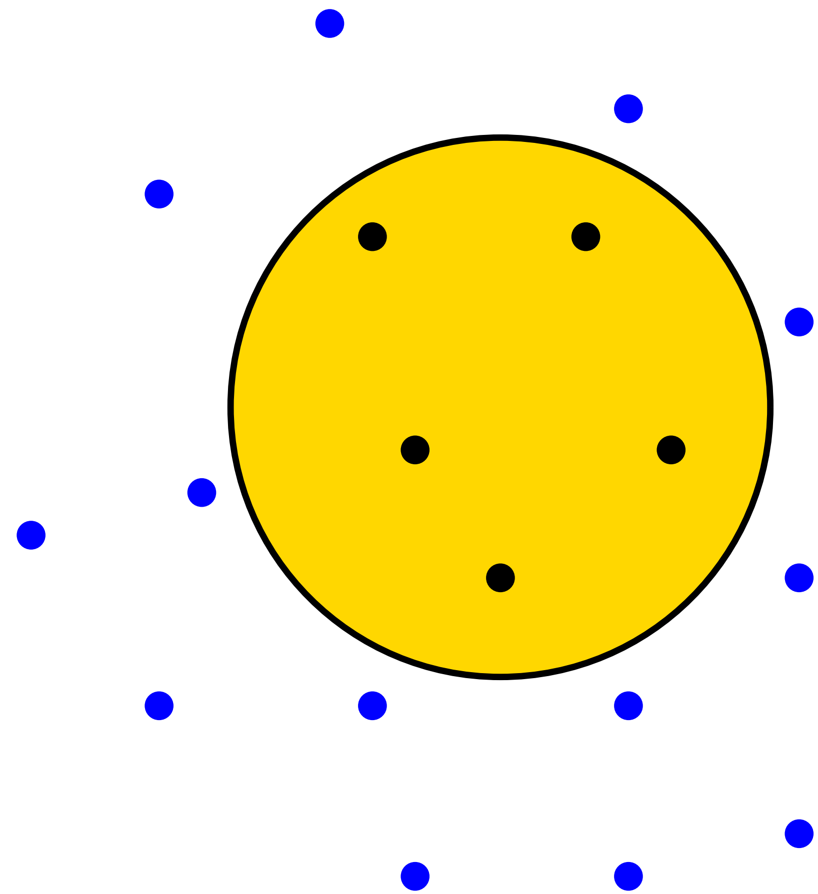

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Find all (x_i, y_i) s.t.,

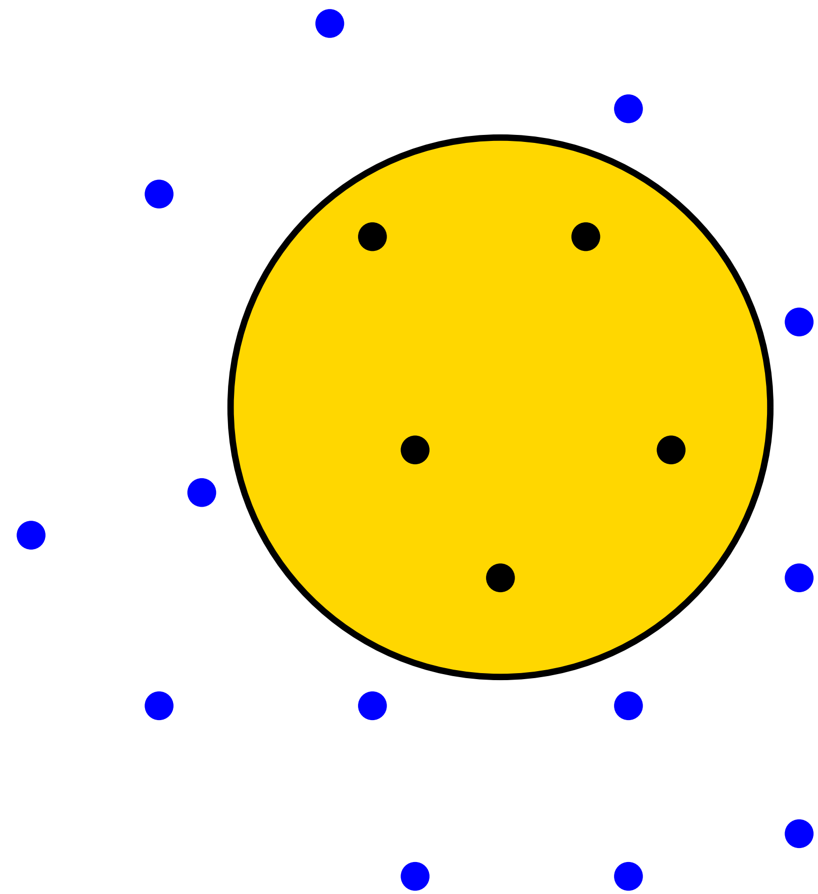
$$(x_i - a)^2 + (y_i - b)^2 \leq r^2$$

$$x_i^2 - 2ax_i + a^2 + y_i^2 - 2by_i + b^2 \leq r^2$$

$$z_i - 2ax_i + a^2 + -2by_i + b^2 \leq r^2$$

$$z_i \leq 2ax_i + 2by_i + r^2 - a^2 - b^2$$

Point $(x_i, y_i, x_i^2 + y_i^2)$ below halfspace
$H(a, b, r) : Z \leq 2aX + 2bY + r^2 - a^2 - b^2$



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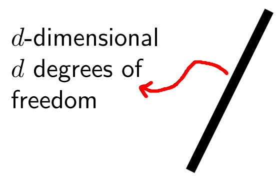

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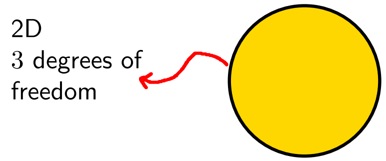

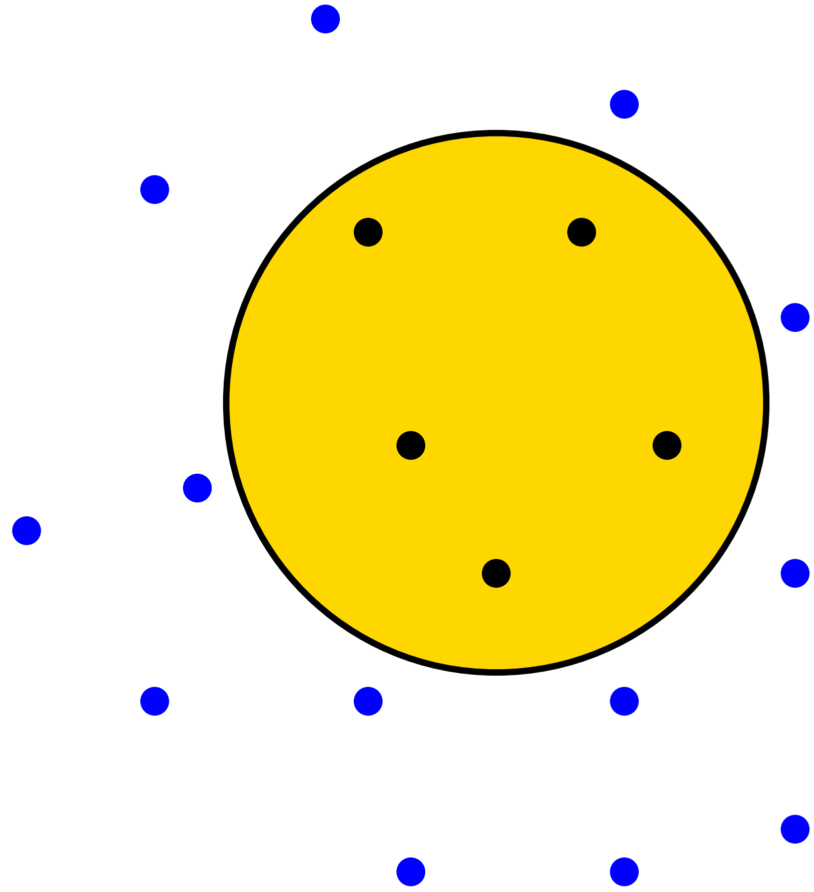

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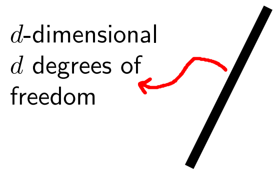

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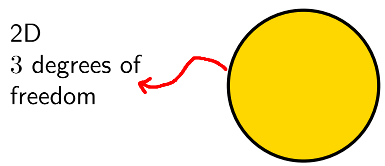
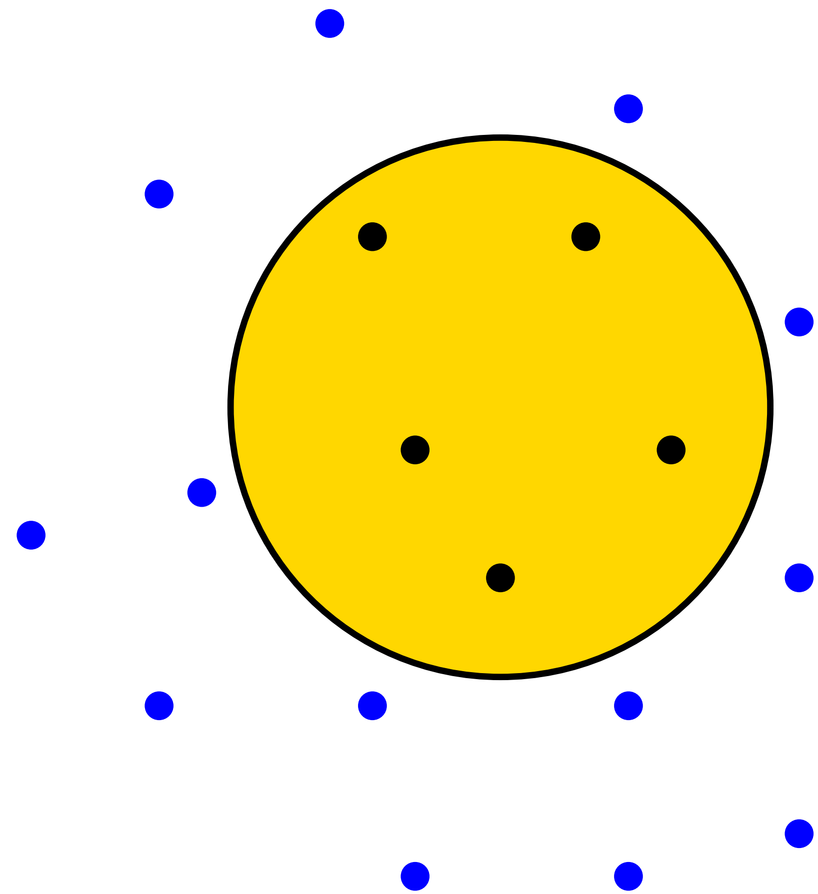


n space, $n^{1-1/3} = n^{2/3}$ query time (low space)

n^3 space, $\log^2 n$ query time (fast query)



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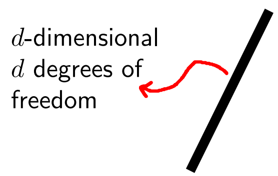

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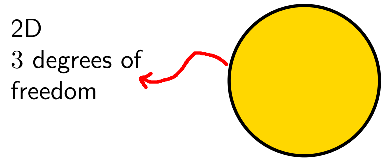

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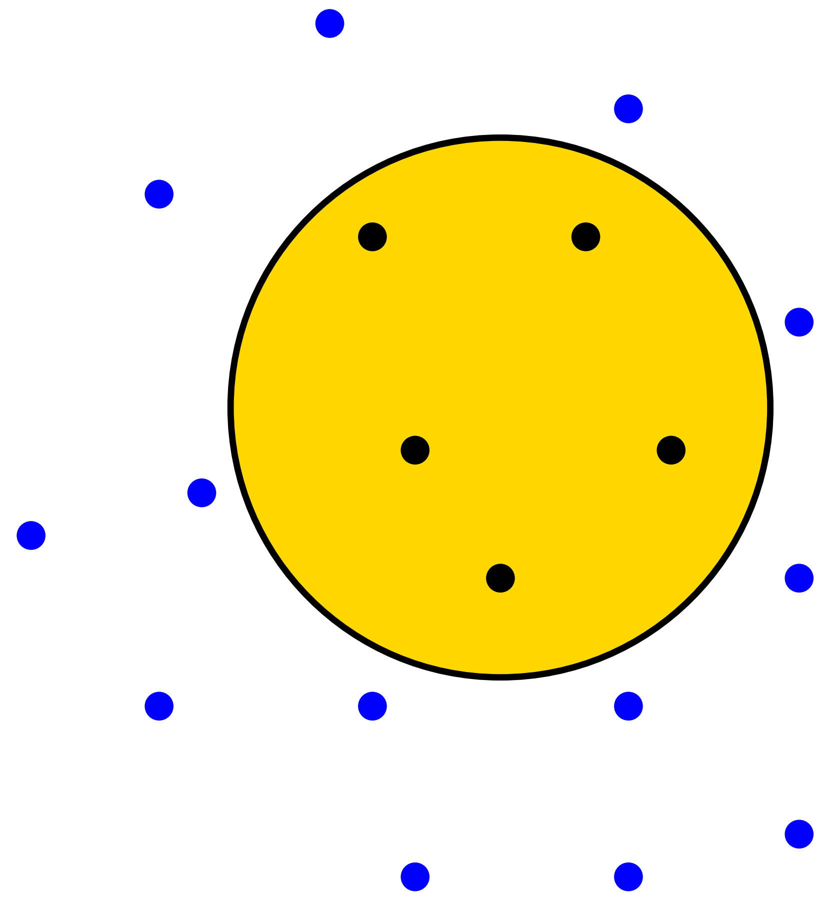
The polynomial
Method



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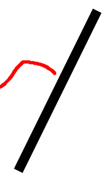
d-dimensional
d degrees of freedom



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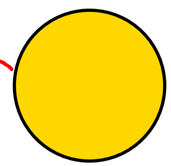
2D
3 degrees of freedom



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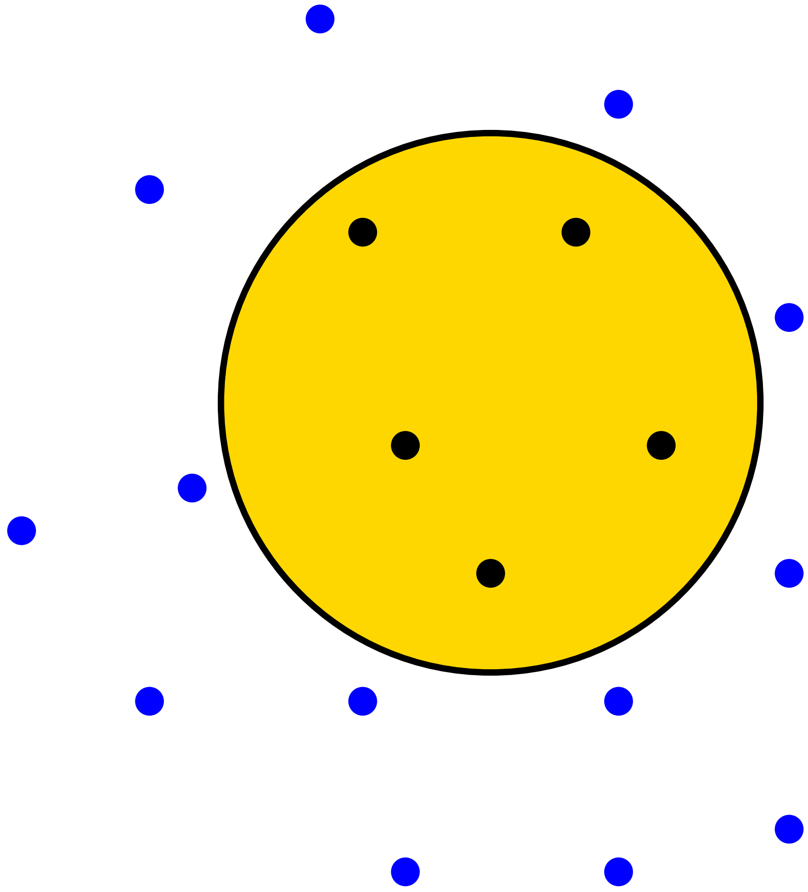
The polynomial Method



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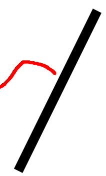

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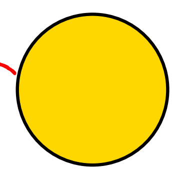

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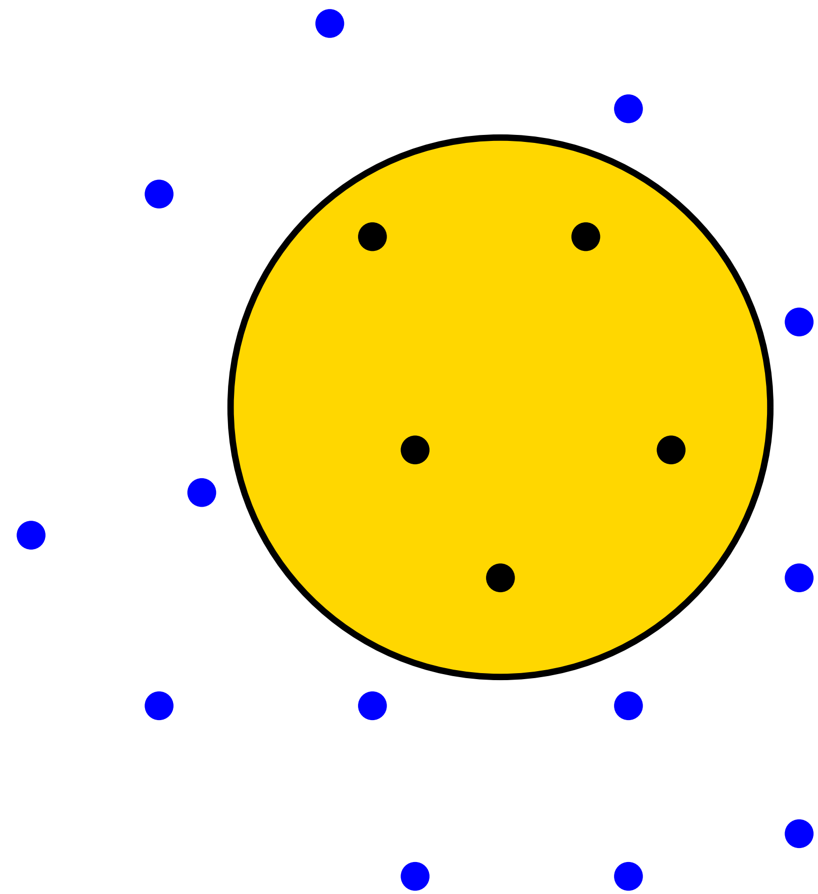

Current knowledge: $\frac{n^3}{Q^5(n)} \leq S(n) \leq \frac{n^3}{Q^4(n)}$

Semialgebraic Range Reporting

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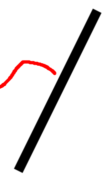

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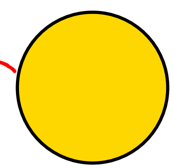

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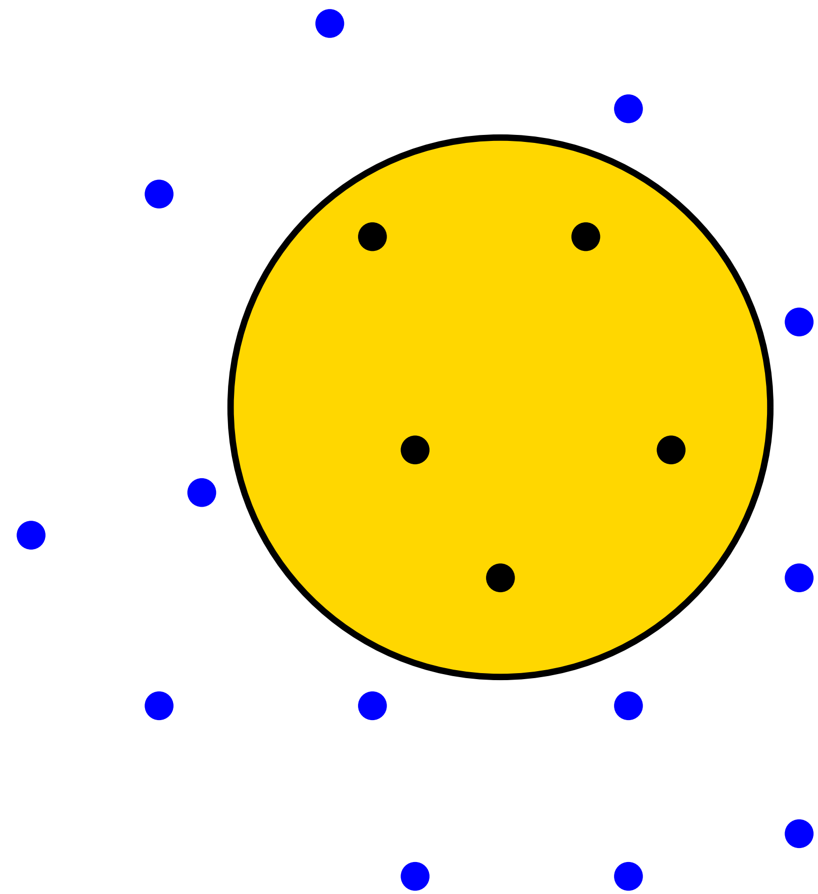
tight

The polynomial Method



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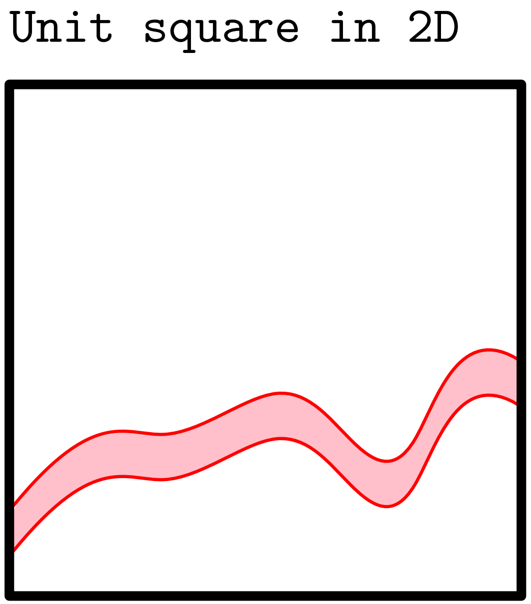
Fast Query Lower Bound: The General Approach



- Input: n uniformly random points
- Query: $-w \leq P(x,y) \leq w$
- List the points in the query
- Goal: Lower bound for polylog $Q(n)$; $Q(n) = \tilde{O}(1)$
- Space Lower Bound: roughly n^{β}
- β: Degrees of freedom

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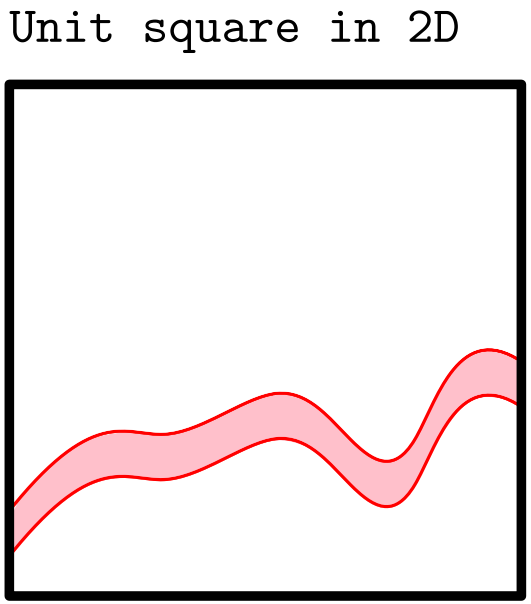

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How to:

- Create n^{β} polynomials $P_i(x, y)$
- Area of $-w \leq P(x, y) \leq w$ is $\Theta(w)$
- $w \approx \frac{Q(n)}{n} = \tilde{O}(1)$: Each region is "$Q(n)$-rich"

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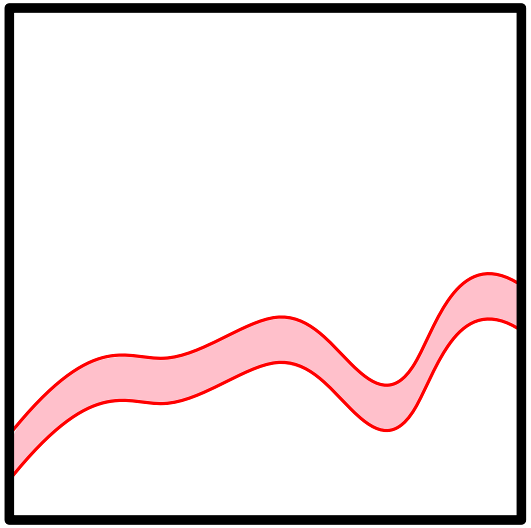

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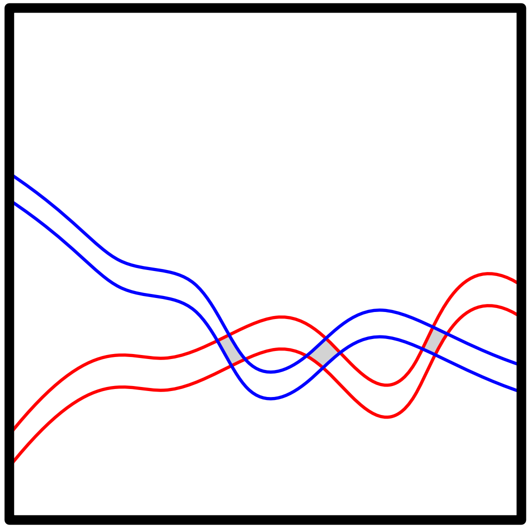
- **(main challenge)** Intersection of two regions: $\ll \frac{1}{n}$

Fast Query Lower Bound: The General Approach

`Unit square in 2D`



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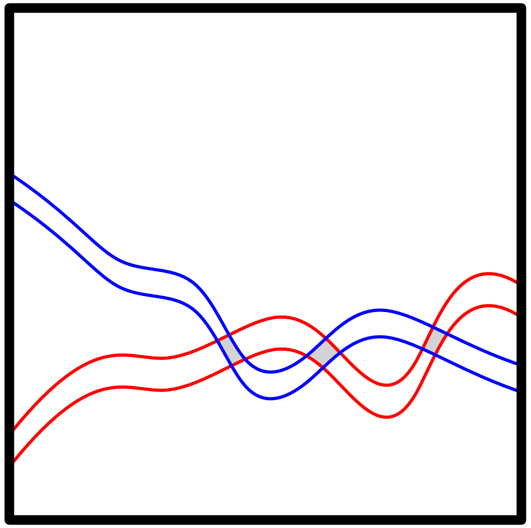

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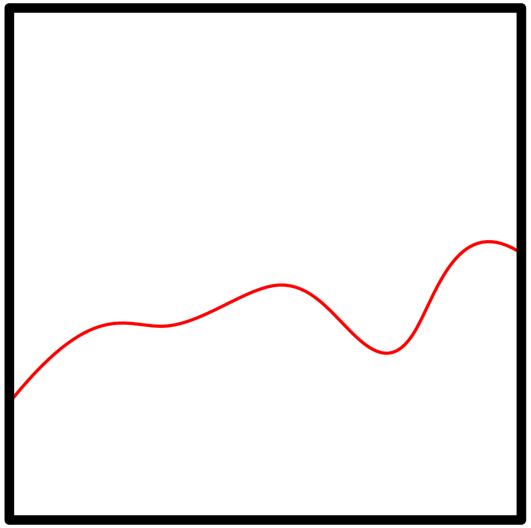
So far only one approach:

Create: $P_1(x,y), P_2(x,y), \ldots, P_M(x,y)$
Min. distance between coefficients is **large**
Prove it implies **(main challenge)**

The First Technique

Unit square in 2D



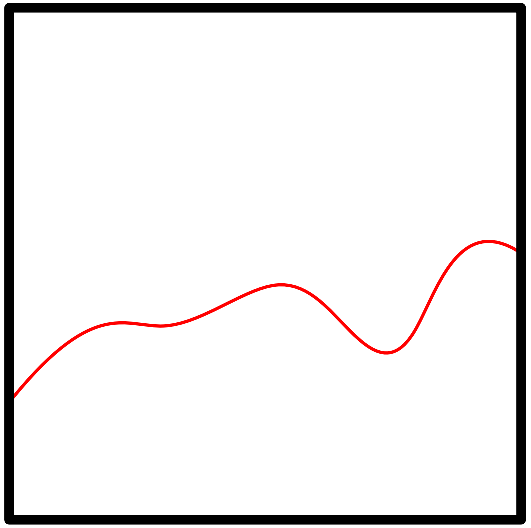
$P(x, y): \quad Y = \square X^{\Delta} + \square X^{\Delta-1} + \ldots + \square X + \square$

How to:

- Create $n^{\Delta+1}$ polynomials $P_i(x, y)$
- $-\frac{Q(n)}{n} \leq P(x, y) \leq \frac{Q(n)}{n}$
- Each region is "$Q(n)$-rich"
- **(main challenge)** Intersection of two regions: $\ll \frac{1}{n}$

The First Technique

Unit square in 2D



$P_j(x,y): \quad Y = \Box X^{\Delta} + \Box X^{\Delta-1} + \ldots + \Box X + \Box$

Distance $\frac{Q^{\Delta}(n)}{n}$ is enough to imply **(main challenge)**

$P_i(x,y): \quad Y = \Box X^{\Delta} + \Box X^{\Delta-1} + \ldots + \Box X + \Box$

How to:

- Create $n^{\Delta+1}$ polynomials $P_i(x,y)$
- $-\frac{Q(n)}{n} \le P(x,y) \le \frac{Q(n)}{n}$
- Each region is "$Q(n)$-rich"
- **(main challenge)** Intersection of two regions: $\ll \frac{1}{n}$

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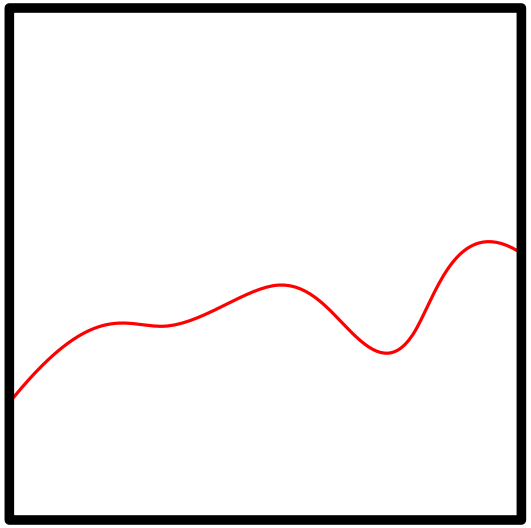

$P(x,y): \quad Y = X^{\Delta} + \square X^{\Delta-1}Y + \ldots + \square X^i Y^j + \ldots + \square Y + \square X + \square$

How to:

- Create $n^{\binom{\Delta+d}{d}}$ polynomials $P_i(x,y)$
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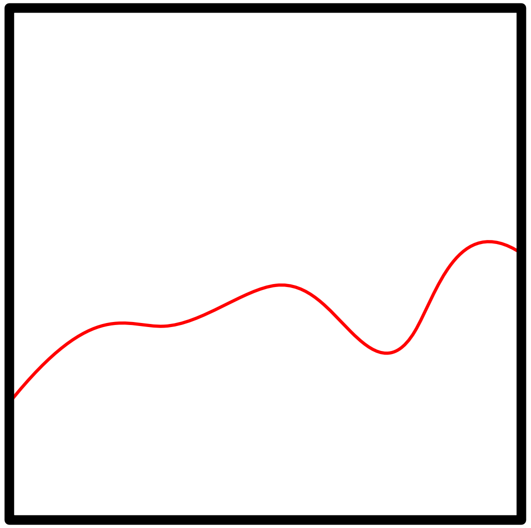

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Distance $\frac{Q^{\Delta^2}(n)}{n}$ and small magnitude is enough to imply **(main challenge)**

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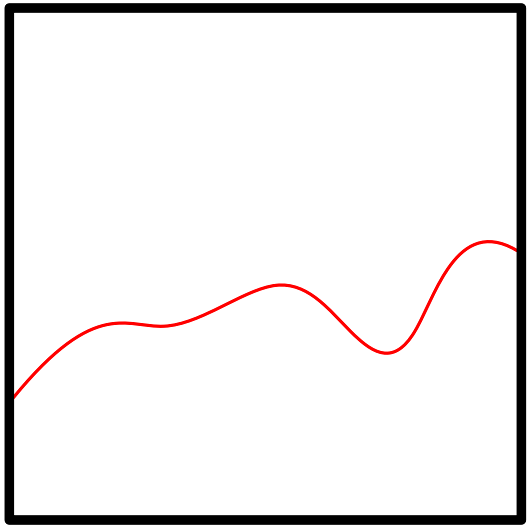
The Main Open Question

Unit square in 2D



$P(x,y): \quad 0 = \square X^{\Delta} + \square X^{\Delta-1} Y + \ldots + \square X^i Y^j + \ldots + \square Y + \square X + \square$

- In many problems, $\square$'s **CANNOT** be independent.
- $\square$ is a polynomial of $a_1, \ldots, a_\beta$
- Some of them have to zero.
- Some of them have to constants
- Some of them depend on other coefficients

How to:

- Create n^β polynomials $P_i(x,y)$
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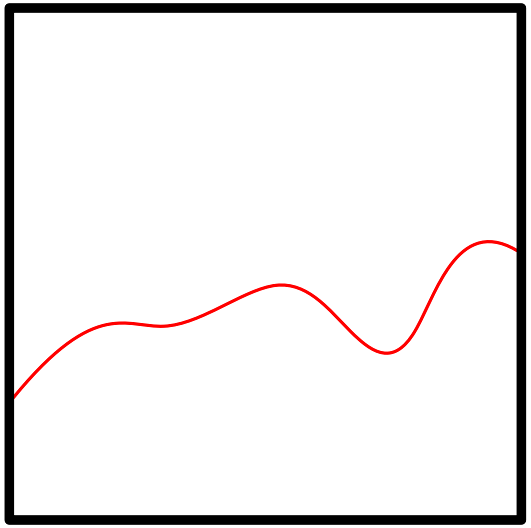

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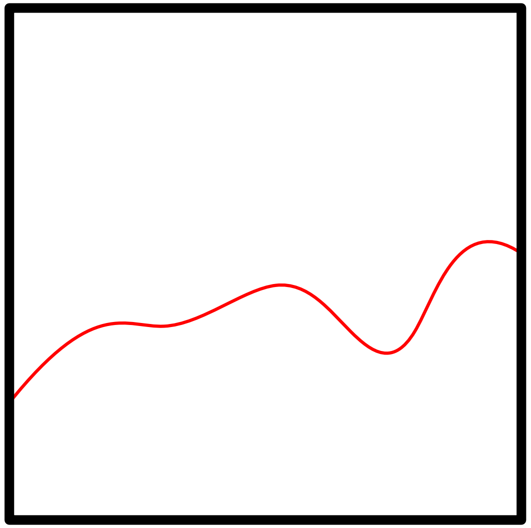
Hurdle:

- $P_1(x,y)H(x,y) = 0$
- $P_2(x,y)H(x,y) = 0$
- Have arbitrary large coefficient distance
- Infinitely many zeroes in common

The Third Technique

Unit square in 2D



$P(x,y): YG(X) = F(X)$
G and F "far from" sharing a root
$YG(X) - F(X)$ is irreducible

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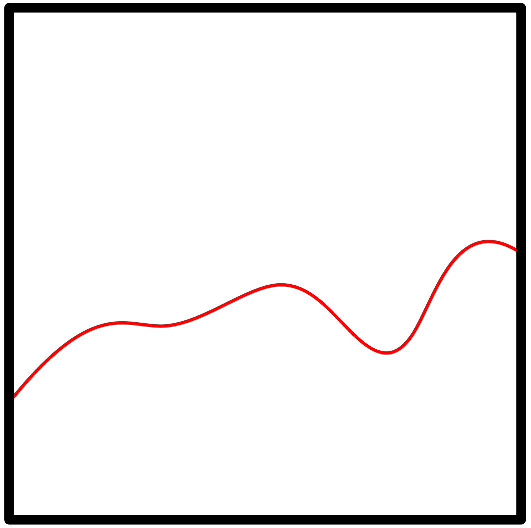

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How to:

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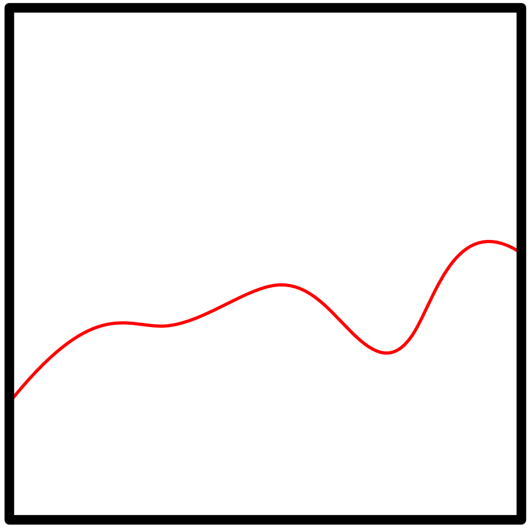
The End?

How to Main Challenge



Setup:

$P(x, y) : YG(X) = F(X)$

$t = \text{Resultant}(F, G) > 0$

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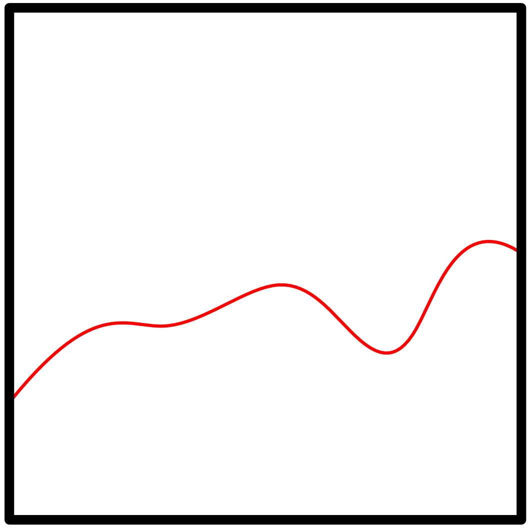

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$P(x, y) : YG(X) = F(X)$
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$$\exists H(X), L(X) : GH + FL = 1$$

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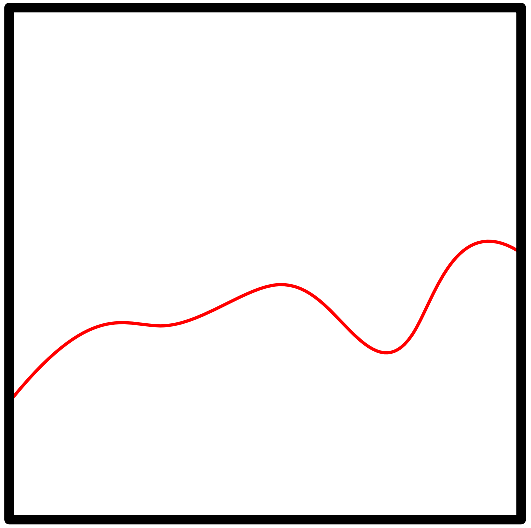

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Create lots of poly:

- A "grid" of side-length δ around P

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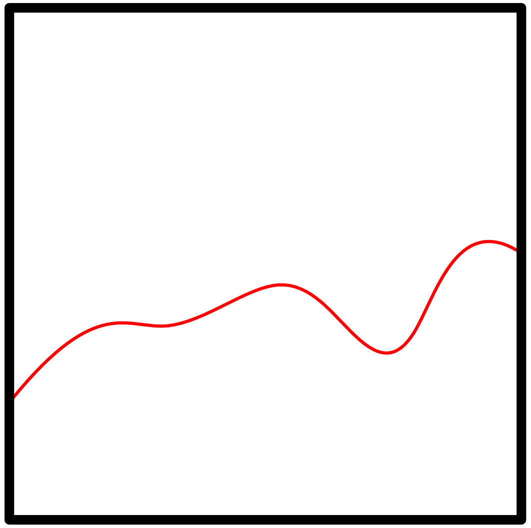

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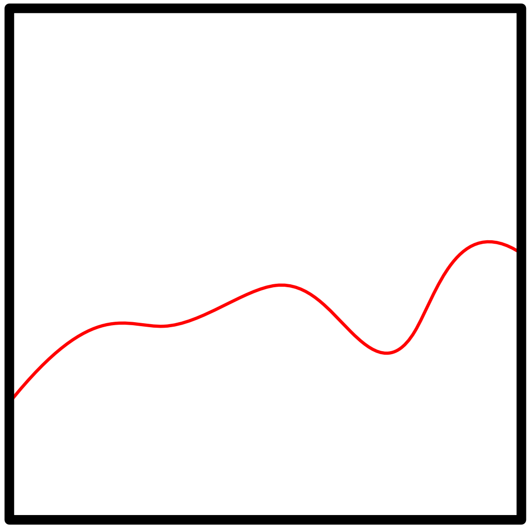

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Get:

- $M = n^\beta$ polys, $P_1, ..., P_M$ (ignoring poly $Q(n)$ factors)
- Every two differ at by at least δ in one coeff.
- Every P_i in a small neighborhood of P (within radius $n\delta$)
- δ sufficiently small constant
- Region: $0 \leq P_i(x, y) \leq \frac{Q(n)}{n} = w$

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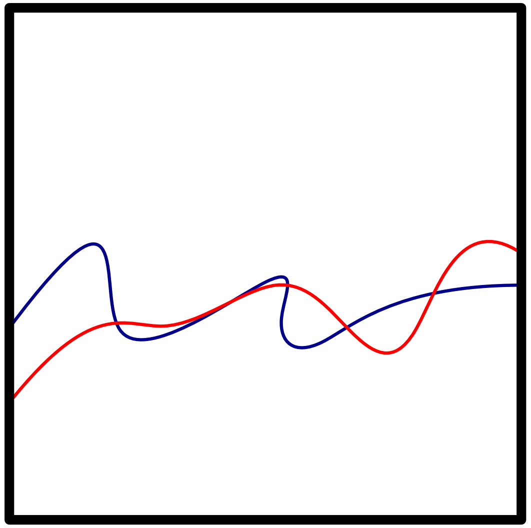

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Consider P_1 and P_2:

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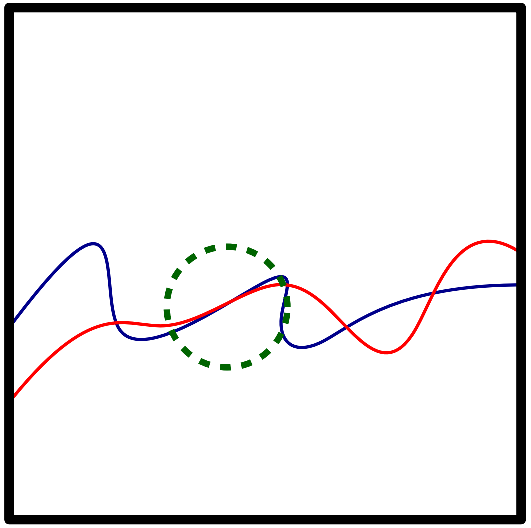

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Imagine big overlap

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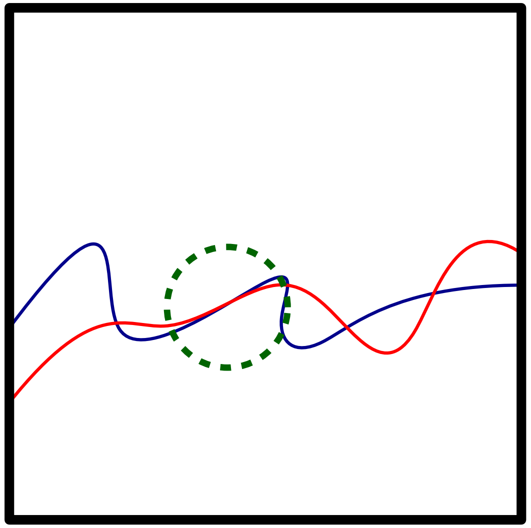

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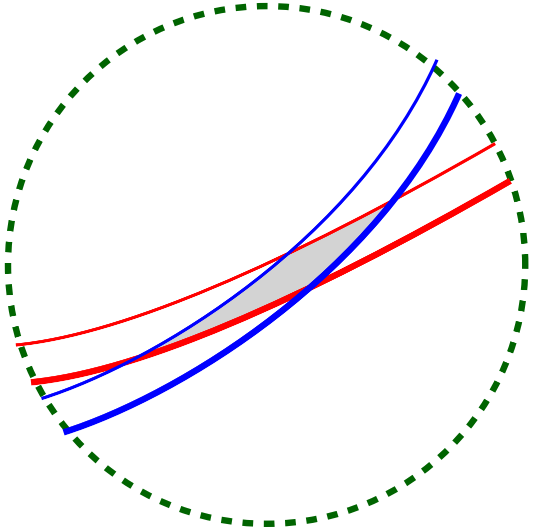

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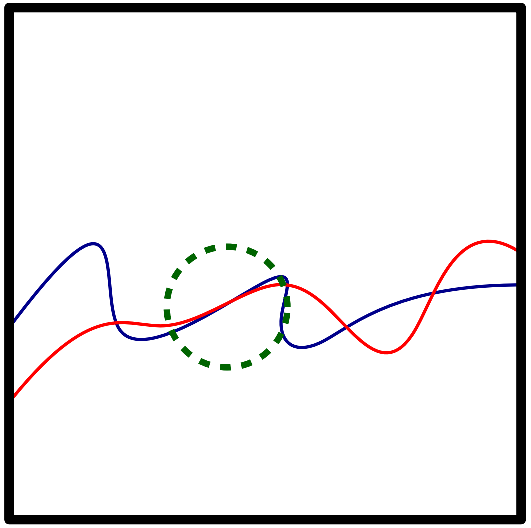

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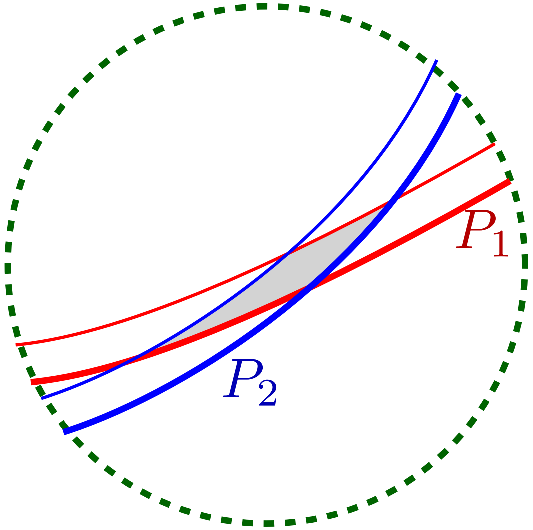

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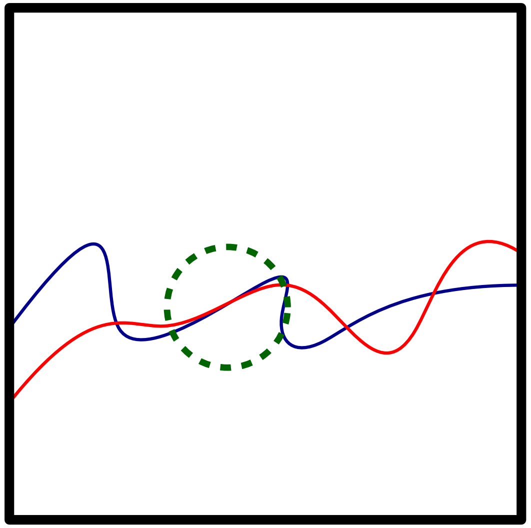

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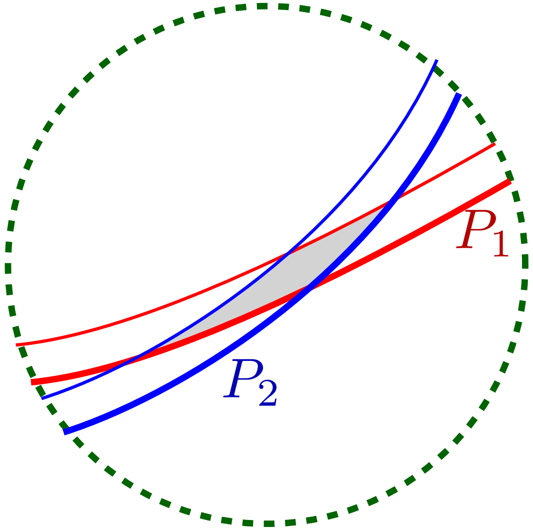
P_1 and P_2 evaluate within $[0, w]$ in a big interval I of length at least $\frac{1}{Q(n)}$



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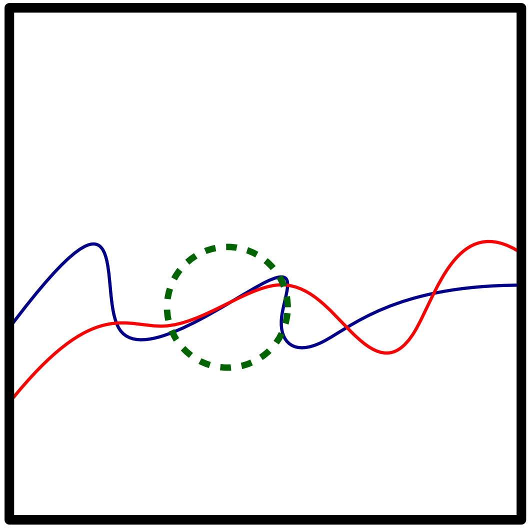

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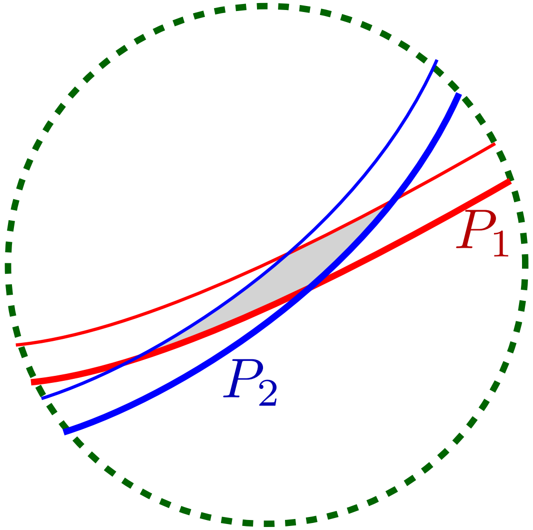

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Approach:

- Pick ℓ points in I on P_1

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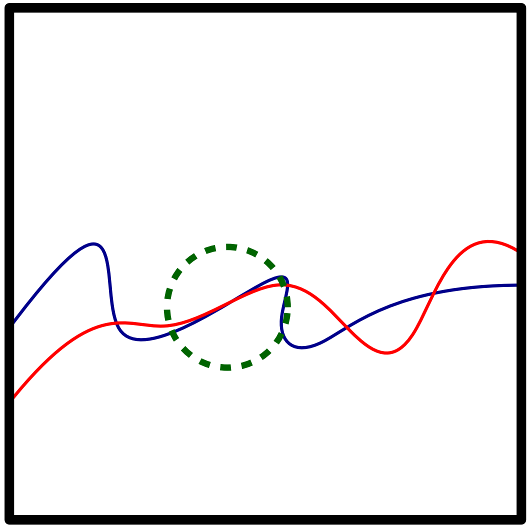

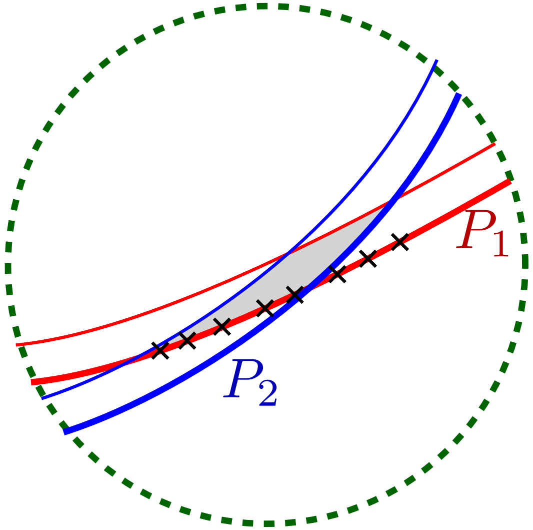

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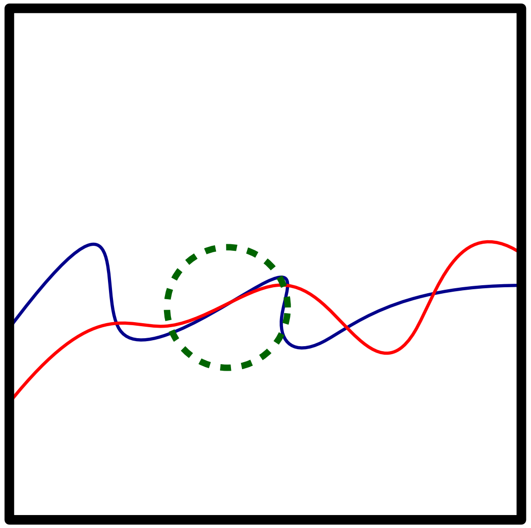

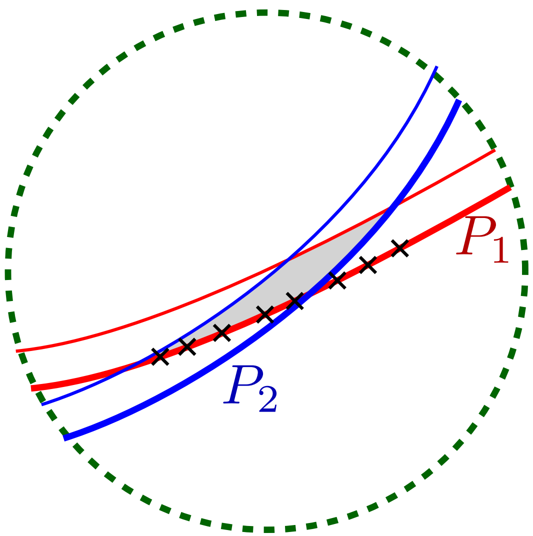

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V: Vector of monomials:

- all monomials except yX^{Δ_G}.
- X^i for $i = 1, ..., k$ so we get ℓ mono. in total
- Build an $\ell \times \ell$ matrix A:
  - Row i is the evaluation of V on the i-th point

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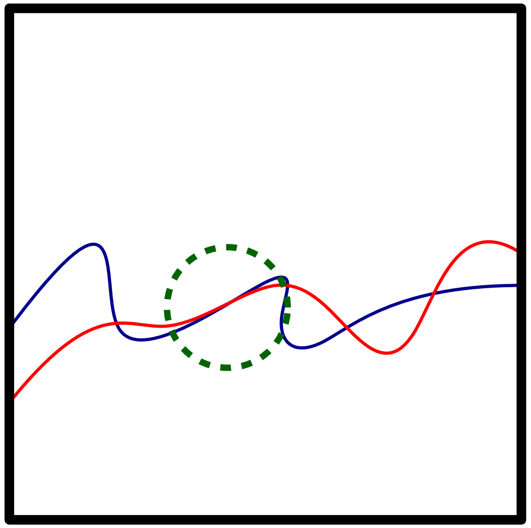

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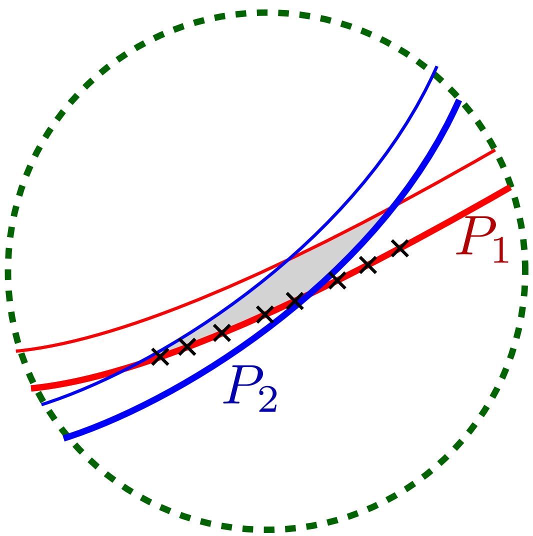

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Claim: $|\det(A)| \geq \text{Resultant}(F, G)|I|^{\ell^2} - O(w)$

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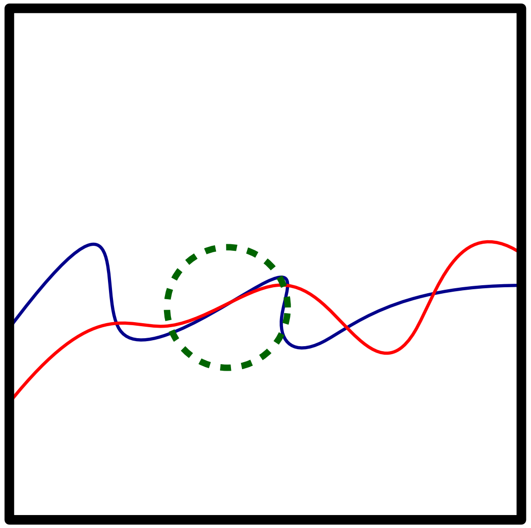

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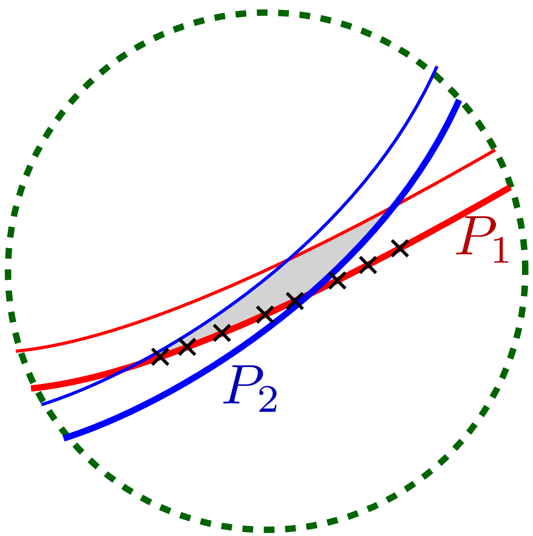

Tweak coeff of P_2 by smaller than δ to pass through the ℓ points $\Rightarrow$ contradiction

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Thank you!