Spaces with nonpositive curvature and their ideal boundaries

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Abstract

We construct a pair of finite piecewise Euclidean 2-complexes with nonpositive curvature which are homeomorphic but whose universal covers have nonhomeomorphic ideal boundaries, settling a question from [8].

1.1 Introduction

The ideal boundary of a locally compact Hadamard space1 $X$ is a compact metrizable space on which the isometry group of $X$ acts by homeomorphisms. Even though the ideal boundary is a well known construct with many applications in the literature (see for example [10, 4, 2]), the action of the isometry group on the boundary has not been studied closely except in the case of symmetric spaces, Gromov hyperbolic spaces, Euclidean buildings, and a handful of other cases. In the Gromov hyperbolic case2 the boundary behaves nicely with respect to quasi-isometries: any quasi-isometry $f : X_1 \to X_2$ between Gromov hyperbolic Hadamard spaces induces a boundary homeomorphism $\partial_\infty f : \partial_\infty X_1 \to \partial_\infty X_2$ [7]. This has the consequence that the ideal boundary is “geometry independent”:

If a finitely generated group $G$ acts discretely, cocompactly and isometrically on two Gromov hyperbolic Hadamard spaces $X_1, X_2$, then there is a $G$-equivariant homeomorphism $\partial_\infty X_1 \to \partial_\infty X_2$.

In [8, p. 136] Gromov asked whether this fundamental property still holds if the hyperbolicity assumption is dropped. Sergei Buyalo [5] and the authors [6] independently answered Gromov’s question negatively: [5, 6] exhibit a pair of deck group invariant Riemannian metrics on a universal cover which have ideal boundaries homeomorphic to $S^2$, such that the deck group actions on the boundaries are topologically

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3 Following [3] we will call complete, simply connected length spaces with nonpositive curvature Hadamard spaces.
4 The same statement is true of higher rank irreducible symmetric spaces and Euclidean buildings by [9].
inequivalent. Gromov also asked if $\partial_\infty X_1$ must be (non-equivariantly) homeomorphic to $\partial_\infty X_2$ whenever $X_1$ and $X_2$ are Hadamard spaces admitting discrete, cocompact, isometric actions by the same finitely generated group $G$. In this paper we show that even this can fail:

**Theorem 1** There is a pair $\tilde{X}_1, \tilde{X}_2$ of homeomorphic finite 2-complexes with non-positive curvature such that the universal covers $X_1, X_2$ have nonhomeomorphic ideal boundaries.

We remark that if $M_1$ and $M_2$ are closed Riemannian manifolds with nonpositive curvature and $\pi_1(M_1) \simeq \pi_1(M_2)$, then their universal covers will have ideal boundaries homeomorphic to spheres of the same dimension.

Although some basic questions about the boundary have now been answered, a number of related issues are wide open, except in a few special cases. It would be interesting to know exactly which geometric features determine the ideal boundary of a Hadamard space up to (equivariant) homeomorphism. This question has a clean answer (see [6]) in the case of graph manifolds or the 2-complexes considered in this paper. In order to answer the question in any generality, it appears that it will be necessary to develop a kind of “generalized symbolic dynamics” for geodesic flows of nonpositively curved spaces.

### 1.2 Notation and preliminaries

A reference for the facts recalled here is [3]. If $X$ is a Hadamard space, then we denote the ideal boundary of $X$ by $\partial_\infty X$, the geodesic segment joining $x_1, x_2 \in X$ by $\overline{x_1x_2}$, and the geodesic ray leaving $p \in X$ in the asymptote class of $\xi \in \partial_\infty X$ by $\overline{p\xi}$. If $p \in X, \xi_1, \xi_2 \in \partial_\infty X$, then $\angle_p(\xi_1, \xi_2)$ is the angle between the initial velocities of the rays $\overline{p\xi_1}, \overline{p\xi_2}$. $\angle_T(\xi_1, \xi_2) := \sup_{p \in X} \angle_p(\xi_1, \xi_2)$ will denote the Tits angle between $\xi_1, \xi_2 \in \partial_\infty X$. If $p \in X$ then $\angle_p(\xi_1, \xi_2) = \angle_T(\xi_1, \xi_2)$ iff the rays $\overline{p\xi_1}$ and $\overline{p\xi_2}$ bound a flat sector.

By the Cartan-Hadamard theorem [1, 3], the universal cover of a connected, complete, length space with nonpositive curvature is a Hadamard space with the natural metric. Let $Z$ be a complete, connected space with nonpositive curvature, and let $\pi : \tilde{Z} \to Z$ be the universal cover. If $Y \subset Z$ is a closed, connected, locally convex subset, then the induced length metric on $Y$ has nonpositive curvature, $\pi^{-1}(Y) \subset \tilde{Z}$ is a disjoint union of closed convex components isometric to $\tilde{Y}$, and the induced map $\pi_1(\tilde{Y}) \to \pi_1(Z)$ is a monomorphism.

### 1.3 Torus complexes

The following piecewise Euclidean 2-complexes were suggested to us by Bernhard Leeb, after a discussion of the graph manifold geometry in [6].

Let $T_0, T_1, T_2$ be flat two-dimensional tori. For $i = 1, 2$, we assume that there are (primitive) closed geodesics $a_i \subset T_0$ and $b_i \subset T_i$ with $\text{length}(a_i) = \text{length}(b_i)$, and we glue $T_i$ to $T_0$ by identifying $a_i$ with $b_i$ isometrically. We assume that $a_1$ and $a_2$ lie in distinct free homotopy classes, and intersect once at an angle $\alpha \in (0, \frac{\pi}{2}]$. The resulting
2-complex $X$ is nonpositively curved as a length space because gluing of nonpositively curved spaces along locally convex subsets produces a nonpositively curved space [3]. We refer to $X$ as a **torus complex**. For $i = 1, 2$ let $\hat{Y}_i := T_0 \cup T_i \subset \hat{X}$. Notice that $\hat{Y}_i$ and $T_0$ are closed, locally convex subsets of $\hat{X}$. Therefore the inclusions $\hat{Y}_i \subset \hat{X}$ and $T_0 \subset \hat{X}$ induce monomorphisms of fundamental groups.

### 1.4 The structure of the universal cover

Let $\pi : X \to \hat{X}$ be the universal covering of $\hat{X}$. $X$ is a Hadamard space by the Cartan-Hadamard theorem. A **block** is a connected component of $\pi^{-1}(\hat{Y}_i) \subset X$, and a **wall** is a connected component of $\pi^{-1}(T_0) \subset X$. Let $B$ and $W$ denote the (locally finite) collection of blocks and walls in $X$. Each block (resp. wall) is a closed, connected, locally convex subset of $X$. Hence by 1.2 each block (resp. wall) is a convex subset of $X$ which is intrinsically isometric to the universal cover of $\hat{Y}_i$ (resp. $T_0$). If $W \in W$, $B \in B$, then either $W \cap B = \emptyset$ or $W \cap B = W$ since $W \cap B$ is open and closed in $W$; $W$ is contained in precisely two blocks, one covering $\hat{Y}_i$ and the other covering $\hat{Y}_i$. If $B_1$, $B_2 \in B$ are distinct blocks and $B_1 \cap B_2 \neq \emptyset$, then (after relabelling if necessary) $B_i$ covers $\hat{Y}_i$ and so $B_1 \cap B_2$ consists of a (convex) union of walls; therefore $B_1 \cap B_2 = W$ for some $W \in W$. When $B_1 \cap B_2 \neq \emptyset$ we will say that the blocks $B_1$ and $B_2$ are adjacent.

$\hat{Y}_i$ is a “flat” $S^1$ bundle over a bouquet of two circles, so the universal cover $Y_i$ of $\hat{Y}_i$ (and hence each block) is isometric to the metric product of a simplicial tree with $R$. A **singular geodesic of a block** $B$ is the inverse image of a vertex under the projection of $B$ to its tree factor. Note that singular geodesics of adjacent blocks which lie in the common wall intersect at angle $\alpha$.

The nerve of $B$ (the simplicial complex recording (multiple) intersections of blocks) is a simplicial tree. (This is just the Bass-Serre tree of the amalgamated free product decomposition $\pi_1(\hat{X}) = \pi_1(\hat{Y}_1) *_{\pi_1(T_0)} \pi_1(\hat{Y}_2).$) To see this note that if $\epsilon > 0$ is sufficiently small and $B_{\epsilon}$ is the collection of (open) $\epsilon$-tubular neighborhoods of blocks, then $\text{Nerve}(B_{\epsilon})$ is isomorphic to $\text{Nerve}(B)$. Using a partition of unity subordinate to this cover of $|\text{Nerve}(B)|$ one gets a continuous map $\phi : X \to |\text{Nerve}(B)|$. Any map $\gamma : S^1 \to |\text{Nerve}(B)|$ can be “lifted” to $X$ up to homotopy: there is a $\hat{\gamma} : S^1 \to X$ so that $p \circ \hat{\gamma}$ is homotopic to $\gamma$. Since $\pi_1(X)$ is trivial, this implies that $\pi_1(|\text{Nerve}(B)|)$ is trivial. In particular, every wall separates $X$. We will say that a wall (resp. block) separates two blocks $B_1$, $B_2 \in B$ if the edge (resp. vertex) of $|\text{Nerve}(B)|$ corresponding to the wall (resp. vertex) lies between the vertices of $|\text{Nerve}(B)|$ corresponding to $B_1$ and $B_2$.

Our plan is to show that the subspace $\cup_{B \in B} \partial_{\infty} B \subset \partial_{\infty} X$ can be characterized purely topologically\(^3\), and that its topology is different depending on whether $\alpha = \frac{\pi}{2}$ or not. It will then follow that a torus complex with $\alpha < \frac{\pi}{2}$ and a torus complex with $\alpha = \frac{\pi}{2}$ have universal covers with nonhomeomorphic ideal boundaries.

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\(^3\)At first glance one might think that $\cup_{B \in B} \partial_{\infty} B$ is a path component of $\partial_{\infty} X$, but this turns out not to be the case. It is a “safe” path component, see 1.7.
1.5 Itineraries

For each $p \in X \setminus \bigcup_{W \in \mathcal{W}} W$, $\xi \in \partial_{\infty} X$, we get a sequence of blocks $B_i$ called the \textit{p-itinerary} (simply the \textit{itinerary} if the basepoint $p$ is understood) of $\xi$, as follows. Let $B_i$ be the $i^{th}$ block that the ray $\overrightarrow{px}$ enters; the ray enters a block $B$ if it reaches a point in $B \setminus \bigcup_{W \in \mathcal{W}} W$. We will denote the \textit{p-itinerary} of $\overrightarrow{px}$ by $\text{Itin}(\overrightarrow{px})$ or $\text{Itin}(\xi)$.

Lemma 2 The itinerary of any $\xi \in \partial_{\infty} X$ is the sequence of successive vertices of a geodesic segment or geodesic ray in the simplicial tree $\text{Nerve}(\mathcal{B})$.

Proof. Blocks are convex, so a geodesic cannot revisit any block which it left. The topological frontier of any $B \in \mathcal{B}$ is the union of the walls contained in $B$, so a geodesic segment which leaves $B$ must arrive at a wall $W \subset B$, and then enter the block $B' \in \mathcal{B}$ adjacent to $B$ along $W$. The collection $\mathcal{B}$ is locally finite, so the lemma follows.

Note that $\xi \in \partial_{\infty} X$ has a finite itinerary iff $\xi \in \partial_{\infty} B$ for some $B \in \mathcal{B}$.

1.6 Local components of $\partial_{\infty} X$

Since each block $B$ is isometric to the product of simplicial tree with $R$, $\partial_{\infty} B$ is homeomorphic to the suspension of a Cantor set. A \textit{pole} of $B$ is one of the two suspension points in $\partial_{\infty} B$.

Lemma 3 If $B_1, B_2 \in \mathcal{B}$, then one of the following holds:

1. $\partial_{\infty} B_1 \cap \partial_{\infty} B_2 = \emptyset$.
2. $B_1 \cap B_2 = W \in \mathcal{W}$ and $\partial_{\infty} B_1 \cap \partial_{\infty} B_2 = \partial_{\infty} W$.
3. There is a $B \in \mathcal{B}$ such that $B \cap B_i = W_i \in \mathcal{W}$ and $\partial_{\infty} B_1 \cap \partial_{\infty} B_2$ is the set of poles of $B$.

Proof. Suppose $B_1, B_2 \in \mathcal{B}$ are distinct blocks, $\xi \in \partial_{\infty} B_1 \cap \partial_{\infty} B_2$, and $W \in \mathcal{W}$ is a wall separating $B_1$ from $B_2$. Choose basepoints $b_i \in B_i$, $w \in W$. If $x_k \in b_i \xi$ is a sequence tending to infinity, and $y_k \in \overline{b_2 \xi}$ is a sequence with $d(y_k, x_k) < C$, then we can find a $z_k \in \overline{x_k y_k} \cap W$ since $W$ separates $B_1$ from $B_2$. Therefore $\overline{wz_k} \subset W$ converges, and the limit ray $\overline{wx}$ lies in $W$. Hence $\xi \in \partial_{\infty} W$.

Note that if $W_1, W_2 \subset B \in \mathcal{B}$, then $\partial_{\infty} W_1 \cap \partial_{\infty} W_2$ is just the set of poles of $B$; and $\xi \in \partial_{\infty} X$ cannot be a pole of two adjacent blocks simultaneously.

The lemma follows, since $\partial_{\infty} B_1 \cap \partial_{\infty} B_2 \neq \emptyset$ now implies that the combinatorial distance between $B_1$ and $B_2$ in $\text{Nerve}(\mathcal{B})$ is at most 2.

Lemma 4 Suppose $\xi$ lies on the ideal boundary of a block $B \in \mathcal{B}$, and assume $\xi$ is not a pole of any block other than $B$. Then the path component of $\xi$ in a suitable neighborhood $\Omega$ of $\xi$ is contained in $\partial_{\infty} B$.

Proof. Case 1: $\xi \in \partial_{\infty} B$ is a pole of $B$. Choose $p \in B \setminus \bigcup_{W \in \mathcal{W}} W$. Recall (see section 1.3) that $\alpha$ is the angle between singular geodesics of adjacent blocks lying in the common wall, so $\alpha$ is the minimum Tits angle between $\xi$ and any pole of a block.
adjacent to $B$. Let $\Omega := \{ \xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \frac{\psi}{2} \}$, where $\angle_p(\xi, \xi')$ is the angle between the initial velocities of the two rays $\overrightarrow{p\xi}, \overrightarrow{p\xi'}$. We define an exit from $B$ to be a singular geodesic $E \subset B$ of a block adjacent to $B$. A ray $\overrightarrow{p\xi}$ exits from $B$ via $E$ if $\overrightarrow{p\xi} \cap B$ is a geodesic segment ending at $E$, and the ray $\overrightarrow{p\xi}$ continues into the block containing $E$. For each exit $E$ from $B$, let $\Omega_E$ be the set of $\xi' \in \Omega$ such that $\overrightarrow{p\xi}$ exits $B$ via $E$.

**Sublemma 5** $\Omega_E$ is an open and closed subset of $\Omega$.

*Proof. Openness.* If $\xi' \in \Omega_E$, then $\overrightarrow{p\xi} \cap B$ is a segment ending at some $e \in E$, and $\overrightarrow{p\xi}$ enters the block $B'$ adjacent to $B$ which contains $E$. But then any sufficiently nearby (in the cone topology) ray $\overrightarrow{p\xi'}$ also leaves $B$ at a point close to $e$; clearly this point must lie on $E$ as the collection of exits is discrete. Therefore $\Omega_E$ is open in $\partial_\infty X$.

*Closedness.* Let $E' \subset E$ be the set of “exit points” for elements of $\Omega_E$: the endpoints of segments $\overrightarrow{p\xi} \cap B$, where $\xi' \in \Omega_E$. $E'$ is bounded, for otherwise we could find a sequence $e_k \in E'$ with $\lim_{k \to \infty} d(e_k, p) = \infty$, and get a limit ray $\overrightarrow{p\xi} \subset B$ with $e_\infty \in \partial_\infty E \subset \partial_\infty B \cap \partial_\infty B'$, and $\angle_p(\xi, e_\infty) \leq \frac{\psi}{2}$; this is absurd since $e_\infty$ is a pole of $B'$ and so $\angle_p(e_\infty, \xi) = \angle_T(e_\infty, \xi) \geq \alpha$. Now suppose $\xi' \in \Omega_E$ and $\lim_{k \to \infty} \xi_k' = \xi' \in \Omega$. We have, after passing to a subsequence if necessary, that $\overrightarrow{p\xi} \cap B = \overrightarrow{p\xi} \cap B$ where $e_k \in E$ and $\lim_{k \to \infty} e_k = e_\infty \in E$. Then $\overrightarrow{p\xi} \cap B$ contains $\overrightarrow{p\xi}$; if $\overrightarrow{p\xi} \cap B \neq \overrightarrow{p\xi}$ then clearly $\overrightarrow{p\xi}$ contains a segment of $E$, forcing $\overrightarrow{p\xi} \subset E$, which contradicts the choice of $p$. Thus we have $\xi' \in \Omega_E$. \(\square\)

It follows that the connected (or path) component of $\xi$ in $\Omega$ is contained in $\partial_\infty B$, since any subset $C \subseteq \Omega$ containing $\xi$ and intersecting $\Omega_E$ admits a separation $C = (C \cap \Omega_E) \cup (C \setminus \Omega_E)$ into open subsets of $C$, and any $\xi' \in \Omega \setminus \partial_\infty B$ lies in $\Omega_E$ for some $E$.

*Case II:* $\xi \in \partial_\infty W$ where $W$ is the wall separating two adjacent blocks $B_1, B_2$, and $\xi$ is not a pole. Pick $p \in W$ not lying on a singular geodesic. Let $\psi$ be the minimum Tits distance between $\xi$ and a pole of $B_i, i = 1, 2$, and set

$$\Omega := \{ \xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \frac{\psi}{2} \}.$$

Let $E$ be a singular geodesic of $B_1$ or $B_2$ which is contained in $W$. We say that the ray $\overrightarrow{p\xi}$ exits $W$ via $E$ if $\overrightarrow{p\xi} \cap W$ ends at a point in $E$, and $\overrightarrow{p\xi}$ then immediately enters the block corresponding to $E$. Let $\Omega_E$ be the set of $\xi' \in \Omega$ so that $\overrightarrow{p\xi}$ exits $W$ via $E$. One checks as in case I that $\Omega_E$ is closed and open in $\Omega$, so we conclude that the connected component of $\xi$ in $\Omega$ is contained in $\partial_\infty W$.

*Case III:* $\xi \in \partial_\infty B$ does not lie in the boundary of any block other than $B$. Let $\phi$ be the minimum Tits angle between $\xi$ and a pole of $B$, and set

$$\Omega := \{ \xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \frac{\phi}{2} \}.$$

Pick $p \in B \setminus \bigcup_{W \in W} W$. Since $\xi$ is not a pole of $B$, the ray $\overrightarrow{p\xi}$ determines an isometrically embedded Euclidean half-plane $H \subset B$, the intersection of the flat planes in $B$ containing it. Let $\mathcal{B}'$ be the collection of blocks adjacent to $B$. If $B' \in \mathcal{B}'$ then
$B' \cap H$ (where $W = B \cap B'$ is the wall between $B$ and $B'$) is either empty, a singular geodesic of $B$, or a flat strip with finite width bounded by singular geodesics, for otherwise we would have $\xi \in \partial_\infty B'$. Removing the singular geodesics and $\cup_{B' \in B} B'$ from $H$, we get a subset $H^0$ whose connected components are a countably infinite collection of open strips. If $S \subset H^0$ is such a strip, we let $\Omega_S$ be the set of $\xi' \in \Omega$ so that $\overline{p\xi'} \cap S \neq \emptyset$. As in cases I and II, $\Omega_S$ is closed and open in $\Omega$. This forces the connected component of $\xi$ in $\Omega$ to be contained in $\partial_\infty H \subset \partial_\infty B$, as desired. \hfill \Box

1.7 Vertices and safe paths

We say that $\xi \in \partial_\infty X$ is a vertex if there is a neighborhood $U$ of $\xi$ such that the path component of $\xi$ in $U$ is homeomorphic to the cone over a Cantor set, with $\xi$ corresponding to the vertex of the cone. By Lemma 4 the set of vertices in $\cup_{B \in B} \partial_\infty B$ is precisely the set of poles in $\cup_{B \in B} \partial_\infty B$ (a priori there may be other vertices in $\partial_\infty X$).

A path $c : [0, 1] \to \partial_\infty X$ is safe if $c(t)$ is a vertex for only finitely many $t \in [0, 1]$. Since being joinable by a safe path is an equivalence relation on pairs of points, and since $\partial_\infty B_1 \cup \partial_\infty B_2$ is safe path connected when $B_1$ is adjacent to $B_2$, it follows that $\cup_{B \in B} \partial_\infty B$ is safe path connected.

Lemma 6 $\cup_{B \in B} \partial_\infty B$ is a safe path component of $\partial_\infty X$.

Proof. First note that if $c : [0, 1] \to \partial_\infty X$ is a path, $c(t)$ is not a vertex when $t \in (0, 1)$, $B \in B$, and $c(0) \notin \partial_\infty B$ is not a pole of any block other than $B$, then $c([0, 1]) \subset \partial_\infty B$. This follows from Lemma 4, the fact that $\partial_\infty B$ is closed in $\partial_\infty X$, and a continuity argument.

Now if $B_0 \in B$, $c : [0, 1] \to \partial_\infty X$ is a safe path starting in $\partial_\infty B_0$, and $0 = t_0 < t_1 < \ldots < t_k = 1$ are chosen so that $c(t)$ is a vertex only if $t = t_i$ for some $i$, then one proves by induction on $i$ that the intervals $[t_{i-1}, t_i]$ are mapped into $\cup_{B \in B} \partial_\infty B$. \hfill \Box

Lemma 7 Pick $B_0 \in B$ and $p \in B_0 \setminus \cup_{W \in W} W$. Let $c : [0, 1] \to \partial_\infty X$ be a path, and suppose $c(0)$ has an infinite p-itinerary. Then either $c(t)$ has the same p-itinerary as $c(0)$ for all $t \in I$, or there is a $t \in I$ so that $c(t)$ has a finite itinerary. In particular, by Lemma 6, if $c$ is a safe path then $c(t)$ has the same p-itinerary as $c(0)$ for all $t \in I$.

Proof. Suppose $\xi_k \in \partial_\infty X$ is a sequence with $\lim_{k \to \infty} \xi_k = \xi \in \partial_\infty X$, and a certain block $B$ is in the itinerary of $p\xi_k$ for every $k$. Then either

1. $Itin(\xi)$ contains $B$

or

2. $Itin(\xi)$ is a sequence with only $B_k$ lying between $B_0$ and $B$.

To see this, suppose $B'$ is in $Itin(\xi)$ and $x \in \overline{p\xi} \cap \text{Int}(B')$. Then $x = \lim_{j \to \infty} x_j$ where $x_j \in \overline{p\xi_j} \cap \text{Int}(B')$ for sufficiently large $j$, so $B'$ is in $Itin(\xi_j)$ for sufficiently large $j$. This means that $B'$ lies between $B_0$ and $B$ for otherwise $B$ would have to lie between $B_0$ and $B'$, forcing $B \in Itin(\xi)$.

The lemma now follows, since if $B$ is in $Itin(c(0))$ but not in $Itin(c(t))$ for all $t \in [0, 1]$, then setting $t_0 := \inf \{ t \mid B \notin Itin(c(t)) \}$ we get a ray $pc(t_0)$ with finite itinerary by the reasoning of the preceding paragraph. \hfill \Box
Corollary 8 There is a unique safe path component of \( \partial \infty X \) which is dense, namely \( \cup_{B \in B} \partial \infty B \).

Proof. By Lemma 6 we know that \( \cup_{B \in B} \partial \infty B \) forms a safe path component. \( \cup_{B \in B} \partial \infty B \) is dense in \( \partial \infty X \) since any initial segment \( \overline{p \xi} \) of a ray \( \overline{p \xi} \) may be continued as a ray \( \overline{p \xi} = \overline{pw} \cup x \overline{\xi} \) where the continuation \( x \overline{\xi} \) lies in a block (one of at most two) containing \( x \).

By Lemma 7, if \( \xi \in \partial \infty X \) has an infinite \( p \)-itinerary, then any safe path starting at \( \xi \) consists of points with the same \( p \)-itinerary. Clearly the collection of points with a given \( p \)-itinerary isn’t dense in \( \partial \infty X \). The corollary follows. \( \square \)

1.8 Detecting block boundaries

Call an arc \( I \subset \cup_{B \in B} \partial \infty B \) an edge if its endpoints are both vertices, but no interior point of \( I \) is vertex of \( \partial \infty X \). Edges are contained in the boundary of a single block \( B \in B \) (see the proof of Lemma 6). Clearly the endpoints of an edge \( I \subset \cup_{B \in B} \partial \infty B \) are either the poles of a single block, or \( I \subset \partial \infty W \) where \( W = B_1 \cap B_2 \) and the endpoints of \( I \) are poles of \( B_1 \) and \( B_2 \). So two points in \( \cup_{B \in B} \partial \infty B \) are the poles of a single block (resp. adjacent blocks) iff they are the endpoints of more than one edge (resp. a unique edge). A subset of \( \cup_{B \in B} \partial \infty B \) is the boundary of a block \( B \in B \) iff it is the union of all edges intersecting the poles of \( B \).

1.9 Limiting behavior of poles

Pick \( B \in B \), and consider the set \( \mathcal{P} \) of poles of blocks adjacent to \( B \). If \( \eta \in \partial \infty B \) is a pole of \( B \), then we have \( \angle_T(\xi, \eta) \in \{ \alpha, \pi - \alpha \} \) for every \( \xi \in \mathcal{P} \). Let \( \mathcal{P}_\alpha := \{ \xi \in \mathcal{P} : \angle_T(\xi, \eta) = \alpha \} \), and \( \mathcal{P}_{\pi - \alpha} := \{ \xi \in \mathcal{P} : \angle_T(\xi, \eta) = \pi - \alpha \} \). Call each arc of \( \partial \infty B \) joining the poles of \( B \) a longitude.

Lemma 9 Each longitude of \( \partial \infty B \) intersects \( \mathcal{P}_\alpha \) (resp. \( \mathcal{P}_{\pi - \alpha} \)) in a single point \( \xi \) with \( \angle_T(\xi, \eta) = \alpha \) (resp \( \angle_T(\xi, \eta) = \pi - \alpha \)).

Proof. Pick \( p \in B, \xi \in \partial \infty B \) with \( \angle_T(\xi, \eta) = \alpha \). Any initial segment \( \overline{p \xi} \) of the ray \( \overline{p \xi} \) may be extended to a segment \( \overline{p y} = \overline{p w} \cup x \overline{\xi} \) so that \( \overline{p y} \cap W = \{ y \} \) for some wall \( W \subset B \). Then \( \overline{p y} \) may be extended as a ray \( \overline{p y} = \overline{p y} \cup y \overline{\xi} \) where \( y \overline{\xi} \subset W \) and \( \xi' \in \mathcal{P}_\alpha \). Therefore \( \xi \in \mathcal{P}_\alpha \). Since \( \angle_T(\cdot, \eta) \) is a continuous function on \( \partial \infty B \), each longitude intersects \( \mathcal{P}_\alpha \) in a single point. Similar reasoning applies to \( \mathcal{P}_{\pi - \alpha} \). \( \square \)

From the lemma we see that any longitude \( l \) of \( \partial \infty B \) intersects \( \mathcal{P} \) in two points if \( \alpha < \frac{\pi}{2} \) and one point if \( \alpha = \frac{\pi}{2} \).

1.10 Distinguishing torus complexes

Let \( \hat{X}_1 \) be a torus complex with \( \alpha < \frac{\pi}{2} \), and let \( \hat{X}_2 \) be a torus complex with \( \alpha = \frac{\pi}{2} \). Let \( X_1 \) and \( X_2 \) be their respective universal covers. A homeomorphism \( f : \partial \infty X_1 \to \partial \infty X_2 \) would carry safe path components to safe path components, block boundaries to block boundaries (Corollary 8 and section 1.8), poles to poles, and longitudes to longitudes. But then section 1.9 gives a contradiction.
References


