Algebraic combinatorics, homework 2.

**Exercise 1.** Let $P$ be a finite poset and $d : P \to P$ a bijection preserving the order (if $x \leq y$ then $f(x) \leq f(y)$). Show that $f^{-1}$ preserves the order. Prove that this last statement is false if we do not assume that $P$ is finite.

**Exercise 2.** For a given $k \geq 1$, let $f_n$ be the number of rooted $k$-ary trees (i.e. trees in which each node has 0 or $k$ children) with $n$ nodes (convention: $f_0 = 1$). We note $f(x) = \sum_{n \geq 0} f_n x^n$. Prove that

$$f(x) = 1 + x f(x)^k.$$  

Give a simple expression for $f_n$.

**Exercise 3.** Let $N_n$ be the set of circular sequences (of 0’s and 1’s) of length $n$. Let $M_d$ be the number of circular sequences of length $d$ that are not periodic.

Prove that

$$|N_n| = \sum_{d|n} |M_d|.$$  

Independently, show that

$$\sum_{d|n} d |M_d| = 2^n.$$  

Conclude that

$$|N_n| = \frac{1}{n} \sum_{d|n} \phi\left(\frac{n}{d}\right) 2^d,$$

where $\phi$ is Euler’s totient function.

**Exercise 4.** Let $L$ be a finite lattice, and $f(a, b)$ be a function from $L^2$ to $\mathbb{R}$. Let

$$F(a, b) = \sum_{c \leq a} f(c, b).$$

Prove that

$$\det (F(a \wedge b, b))_{a,b \in L} = \prod_{x \in L} f(x, x).$$

Deduce that

$$\det(gcd(i, j))_{i,j=1}^n = \prod_{k=1}^n \phi(k).$$

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1 More precisely, if

$$\tau_n : \{0,1\}^n \to \{0,1\}^n \quad \text{ with } \quad \tau_n : (a_1, a_2, \ldots, a_n) \mapsto (a_n, a_1, \ldots, a_{n-1})$$

then $a$ and $b$ in $\{0,1\}^n$ are considered to be the same element in $N_n$ if $\tau^k_n a = b$ for some $k \geq 0$.

2 In other words, $M_d$ is the set of elements $a \in \{0,1\}^d$, identified up to the above shift, such that $\tau^d_n a = a$ implies $d \mid k$
Exercise 5. Let $N_d$ be the number of monic irreducible polynomials of degree $d$ over the finite field $\mathbb{F}_q$ with $q$ elements. Prove that
\[
\frac{1}{1 - qx} = \prod_{d=1}^{\infty} \left( \frac{1}{1 - x^d} \right)^{N_d}.
\]
Conclude that $\frac{q^n}{n} = \sum_{d|n} N_d \frac{1}{n/d}$, and
\[
N_d = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) q^d.
\]