Algebraic combinatorics, homework 5.

Exercise 1. Prove that the number of standard Young tableaux of shape \((n,n)\) equals the Catalan number \(C_n\). Prove that the number of permutations \(\pi \in S_n\) with longest decreasing subsequence of length at most two equals the Catalan number \(C_n\).

Exercise 2. We abbreviate \(e_k(n)\) for \(e_k(x_1,\ldots,x_n)\), and similarly for the complete symmetric functions \(h_k\). Prove that
\[
e_k(n) = e_k(n-1) + x_n e_{k-1}(n-1), \quad h_k(n) = h_k(n-1) + x_n h_{k-1}(n).
\]
The Stirling number of the first kind \(c_{n,k}\) can be defined as the number of elements in \(S_n\) with \(k\) disjoint cycles. The Stirling number of the second kind \(S_{n,k}\) can be defined as the number of partitions of the set \([1,n]\) into \(k\) subsets. Prove that
\[
c_{n,k} = c_{n-1,k-1} + (n-1) c_{n-1,k}, \quad S_{n,k} = S_{n-1,k-1} + k S_{n-1,k}.
\]

Exercise 3. Prove the following determinantal identity:
\[
\det((x_i^a \cdots x_i^b)(x_j^a \cdots x_j^b))_{i,j}^n = \prod_{1 \leq i < j \leq n} (x_i - x_j) \prod_{2 \leq i \leq j \leq n} (b_i - a_j).
\]

Exercise 4. Prove that \(s_n(x_1,\ldots,x_n) = h_n(x_1,\ldots,x_n)\).

Exercise 5. Prove that any character of \(S_n\) is an integer-valued function.

Exercise 6. Let \(p_k(x_1,\ldots,x_r) = \sum_{j=1}^r x_j^k\), and for \(\lambda = (\lambda_1,\ldots,\lambda_r) \in \Par(m)\), \(p_\lambda = \prod_{i=1}^r p_{\lambda_i}\). Prove that, in the language of Pólya’s theory,
\[
e_n(x_1,\ldots,x_r) = Z(S_n;p_1,-p_2,\ldots,(-1)^{n-1}p_n).
\]
Prove that \(\{p_\lambda, \lambda \in \Par(m)\}\) is a basis for \(\Lambda^m(X)\).

Exercise 7. Let \(\alpha = (\alpha_1,\ldots,\alpha_n), \beta = (\beta_1,\ldots,\beta_n)\). Prove the Giambelli identity,
\[
s_{(\alpha\mid\beta)} = \det(s_{(\alpha\mid\beta_i)})_{i,j}^n,
\]
where we use the Frobenius notation for partitions.