Probability, homework 4.

Exercise 1.
(a) Let \((f_n)_{n \geq 0}\) be a sequence of nonnegative functions converging almost surely (for the Lebesgue measure \(d\mu\)) to \(f\). Assume that \(\int f_n d\mu \to c < \infty\) as \(n \to \infty\). Prove that \(\int f d\mu\) is defined in \([0, c]\), but does not have to be necessarily \(c\).

(b) Build a sequence of functions \((f_n)_{n \geq 0}\), \(0 \leq f_n \leq 1\), such that \(\int f_n d\mu \to 0\) but for any \(x \in \mathbb{R}\), \((f_n(x))_{n \geq 0}\) does not converge.

Exercise 2. Let \((d_n)_{n \geq 0}\) be a sequence in \((0, 1)\), and \(K_0 = [0, 1]\). We define iteratively \((K_n)_{n \geq 0}\) in the following way. From \(K_n\), which is the union of closed disjoint intervals, we define \(K_{n+1}\) by removing from each interval of \(K_n\) an open interval, centered at the middle of the previous one, with length \(d_n\) times the length of the previous one. Let \(K = \bigcap_{n \geq 0} K_n\) (\(K\) is called a Cantor set).

(a) Prove that \(K\) is an uncountable compact set, with empty interior, and whose points are all accumulation points.

(b) What is the Lebesgue measure of \(K\)?

Exercise 3. On a probability space \((\Omega, \mathcal{A}, P)\) is given a random variable \((X, Y)\) with values in \(\mathbb{R}^2\).

(a) If the law of \((X, Y)\) is \(\lambda \mu e^{-\lambda x - \mu y} 1_{\mathbb{R}^2} (x, y) dx dy\), what is the law of \(\min(X, Y)\)?

(b) If the law of \((X, Y)\) is \(\frac{1}{2\pi} e^{-\frac{x^2 + y^2}{2}} dx dy\), what is the law of \(X/Y\)?

Exercise 4. Let \(\alpha > 0\) and, given \((\Omega, \mathcal{A}, P)\), let \((X_n, n \geq 1)\) be a sequence of independent real random variables with law \(P(X_n = 1) = \frac{1}{n^\alpha}\) and \(P(X_n = 0) = 1 - \frac{1}{n^\alpha}\). Prove that \(X_n \to 0\) in \(L^1\), but that almost surely

\[
\limsup_{n \to \infty} X_n = \begin{cases} 
1 & \text{if } \alpha \leq 1 \\
0 & \text{if } \alpha > 1 
\end{cases}.
\]

Exercise 5. You toss a coin repeatedly and independently. The probability to get a head is \(p\), a tail is \(1 - p\). Let \(A_k\) be the following event: \(k\) or more consecutive heads occur amongst the tosses numbered \(2^k, \ldots, 2^{k+1} - 1\). Prove that \(P(A_k \text{ i.o.}) = 1\) if \(p \geq 1/2\), 0 otherwise.

Exercise 6. Let \(\epsilon > 0\) and \(X\) be uniformly distributed on \([0, 1]\). Prove that, almost surely, there exists only a finite number of rationals \(\frac{p}{q}\), with \(p \wedge q = 1\), such that

\[
\left| X - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}.
\]