Exercise 1. Let $X$ be a standard Gaussian random variable. What is the density of $1/X^2$?

Exercise 2. In the $(O,x,y)$ plane, a random ray emerges from a light source at point $(-1,0)$, towards the $(O,y)$ axis. The angle with the $(O,x)$ axis is uniform on $(-\frac{\pi}{2}, \frac{\pi}{2})$. What is the distribution of the impact point with the $(O,y)$ axis?

Exercise 3. Let $f$ be a continuous function on $[0,1]$. Calculate the asymptotics, as $n \to \infty$, of 
$$\int_{[0,1]^n} f\left(\frac{x_1 + \cdots + x_n}{n}\right) dx_1 \cdots dx_n.$$ 

Exercise 4. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the $L^\infty$ norm (this is the Bernstein-Weierstrass theorem). Let $f$ be a continuous function on $[0,1]$. The $n$-th Bernstein polynomial is 
$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

a) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters $n$ and $x$: $B^{(n,x)} = \sum_{\ell=1}^{n} X_\ell$ where the $X_\ell$'s are independent and $P(X_\ell = 1) = x$, $P(X_\ell = 0) = 1-x$. Prove that $B_n(x) = E(f(S_n(x)))$.

b) Prove that $\|B_n - f\|_{L^\infty([0,1])} \to 0$ as $n \to \infty$.

Exercise 5. The problem of the collector. Let $(X_k)_{k \geq 1}$ be a sequence of independent random variables uniformly distributed on $\{1, \ldots, n\}$. Let $\tau_n = \inf\{m \geq 1 : \{X_1, \ldots, X_m\} = \{1, \ldots, n\}\}$ be the first time for which all values have been observed.

a) Let $\tau_n^{(k)} = \inf\{m \geq 1 : |\{X_1, \ldots, X_m\}| = k\}$. Prove that the random variables 
$$(\tau_n^{(k)} - \tau_n^{(k-1)})_{2 \leq k \leq n}$$
are independent and calculate their respective distributions.

b) Deduce that $\frac{\tau_n}{n \log n} \to 1$ in probability as $n \to \infty$, i.e. for any $\varepsilon > 0$, 
$$P\left(\left|\frac{\tau_n}{n \log n} - 1\right| > \varepsilon\right) \to 0.$$ 

Exercise 6. For any $d \geq 1$, we admit that there is only one probability measure $\mu$ on $S_d$, (the $(d-1)$-th dimensional sphere embedded in $\mathbb{R}^d$) that is uniform, in the following sense: for any isometry $A \in O(d)$ (the orthogonal group in $\mathbb{R}^d$), and any continuous function $f : S_d \to \mathbb{R}$, 
$$\int_{S_d} f(x) d\mu(x) = \int_{S_d} f(Ax) d\mu(x).$$
Let \( X = (X_1, \ldots, X_d) \) be a vector of independent centered and reduced Gaussian random variables.

a) Prove that the random variable \( U = X/\|X\|_{L^2} \) is uniformly distributed on the sphere.

b) Prove that, as \( d \to \infty \), the main part of the globe is concentrated close to the Equator, i.e. for any \( \varepsilon > 0 \),

\[
\int_{x \in S^d_+ \setminus |x_1| < \varepsilon} d\mu(x) \to 1.
\]