Exercise 1. Let \((X_n)_{n \geq 1}\) be independent such that \(\mathbb{E}(X_i) = m_i\), \(\text{var}(X_i) = \sigma_i^2\), \(i \geq 1\). Let \(S_n = \sum_{i=1}^{n} X_i\) and \(\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)\).

a) Find sequences \((b_n)_{n \geq 1}\), \((c_n)_{n \geq 1}\) of real numbers such that \((S_n^2 + b_nS_n + c_n)_{n \geq 1}\) is a \((\mathcal{F}_n)_{n \geq 1}\)-martingale.

b) Assume moreover that there is a real number \(\alpha > 0\) such that \(e^{\lambda X_i} \in L^1\) for any \(i \geq 1\). Find a sequence \((a_n(\lambda))_{n \geq 1}\) such that \((e^{\lambda S_n - a_n(\lambda)})_{n \geq 1}\) is a \((\mathcal{F}_n)_{n \geq 1}\)-martingale.

Exercise 2. Let \((S_n)_{n \geq 0}\) be a \((\mathcal{F}_n)\)-martingale and \(\tau\) a stopping time with finite expectation. Assume that there is a \(c > 0\) such that, for all \(n\), \(\mathbb{E}(|S_{n+1} - S_n| \mid \mathcal{F}_n) < c\).

Prove that \((S_{\tau \wedge n})_{n \geq 0}\) is a uniformly bounded martingale, and that \(\mathbb{E}(S_\tau) = \mathbb{E}(S_0)\).

Consider now the random walk \(S_n = \sum_{k=1}^{n} X_k\), the \(X_k\)'s being iid, \(\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2\). For some \(a \in \mathbb{N}^*\), let \(\tau = \inf\{n \mid S_n = -a\}\). Prove that \(\mathbb{E}(\tau) = \infty\).

Exercise 3. As previously, consider the random walk \(S_n = \sum_{k=1}^{n} X_k\), the \(X_k\)'s being iid, \(\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2\), \(\mathcal{F}_n = \sigma(X_i, 0 \leq i \leq n)\).

Prove that \((S_n^2 - n, n \geq 0)\) is a \((\mathcal{F}_n)\)-martingale. Let \(\tau\) be a bounded stopping time. Prove that \(\mathbb{E}(S_\tau^2) = \mathbb{E}(\tau)\).

Take now \(\tau = \inf\{n \mid S_n \in \{-a, b\}\}\), where \(a, b \in \mathbb{N}^*\). Prove that \(\mathbb{E}(S_\tau) = 0\) and \(\mathbb{E}(S_\tau^2) = \mathbb{E}(\tau)\). What is \(\mathbb{P}(S_\tau = -a)\)? What is \(\mathbb{E}(\tau)\)? Get the last result of the previous exercise by justifying the limit \(b \to \infty\).

Exercise 4. Let \(X_n, n \geq 0\), be iid complex random variables such that \(\mathbb{E}(X_1) = 0, 0 < \mathbb{E}(|X_1|^2) < \infty\). For some parameter \(\alpha > 0\), let

\[
S_n = \sum_{k=1}^{n} \frac{X_k}{k^\alpha}.
\]

Prove that if \(\alpha > 1/2\), \(S_n\) converges almost surely. What if \(0 < \alpha \leq 1/2\) ?

Exercise 5. In a game between a gambler and a croupier, suppose that the total capital in play is 1. After the \(n\)th hand the proportion of the capital held by the gambler is denoted \(X_n \in [0, 1]\), thus that held by the croupier is \(1 - X_n\). We assume \(X_0 = p \in (0, 1)\). The rules of the game are such that after \(n\) hands, the probability for the gambler to win the \((n + 1)\)th hand is \(X_n\); if he does, he gains half of the capital the croupier held after the \(n\)th hand, while if he loses he gives half of his capital. Let \(\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)\).

a) Show that \((X_n)_{n \geq 0}\) is a \((\mathcal{F}_n)_{n \geq 0}\) martingale.

b) Show that \((X_n)_{n \geq 1}\) converges a.s. and in \(L^2\) towards a limit \(Z\).

c) Show that \(\mathbb{E}(X_{n+1}^2) = \mathbb{E}(3X_n^2 + X_n)/4\). Deduce that \(\mathbb{E}(Z^2) = \mathbb{E}(Z) = p\). What is the law of \(Z\)?

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d) For any \( n \geq 0 \), let \( Y_n = 2X_{n+1} - X_n \). Find the conditional law of \( X_{n+1} \) knowing \( \mathcal{F}_n \). Prove that \( \mathbb{P}(Y_n = 0 \mid \mathcal{F}_n) = 1 - X_n \), \( \mathbb{P}(Y_n = 1 \mid \mathcal{F}_n) = X_n \) and express the law of \( Y_n \).

e) Let \( G_n = \{Y_n = 1\} \), \( P_n = \{Y_n = 0\} \). Prove that \( Y_n \to Z \) a.s. and deduce that \( \mathbb{P}(\lim \inf_{n \to \infty} G_n) = p \), \( \mathbb{P}(\lim \inf_{n \to \infty} P_n) = 1 - p \). Are the variables \( \{Y_n, n \geq 1\} \) independent?

f) Interpret the questions c), d), e) in terms of gain, loss, for the gambler.

Exercise 6. Let \( a > 0 \) be fixed, \( (X_i)_{i \geq 1} \) be iid, \( \mathbb{R}^d \)-valued random variables, uniformly distributed on the ball \( B(0, a) \). Set \( S_n = x + \sum_{i=1}^n X_i \).

a) Let \( f \) be a superharmonic function. Show that \( (f(S_n))_{n \geq 1} \) defines a supermartingale.

b) Prove that if \( d \leq 2 \) any nonnegative superharmonic function is constant. Does this result remain true when \( d \geq 2 \)?