Probability, midterm.

Nine exercises perfectly solved yield the maximal possible grade. You therefore should read all of them first.

**Exercise 1.** Let $(\Omega, A, \mathbb{P})$ be a probability space. Prove that if $A \cap B = \emptyset$ and $A, B$ are independent, then $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

**Exercise 2.** Define an algebra and a $\sigma$-algebra. Let $\Omega$ be an infinite set (countable or not). Let $A$ be the set of subsets of $\Omega$ that are either finite or with finite complement in $\Omega$. Prove that $A$ is an algebra but not a $\sigma$-algebra.

**Exercise 3.** Let $X$ be a Poisson random variable with parameter $\lambda$. Prove that for any $r \in \mathbb{N}^*$, $\mathbb{E}(X(X-1) \ldots (X-r+1)) = \lambda^r$. What is the variance of $X$?

**Exercise 4.** Let $X$ be a positive random variable with density $f$. What is the density of $1/(1+X)$?

**Exercise 5.** Let $X$ be a standard Gaussian random variable. Prove that for any $n \in \mathbb{N}^*$, $\mathbb{E}(X^{2n+1}) = 0$ and $\mathbb{E}(X^{2n}) = \frac{(2n)!}{2^n n!}$. You could for example use an expansion of the characteristic function of $X$.

**Exercise 6.** Let $S_n = \sum_{k=1}^n X_1$ where the $X_1$’s are i.i.d. and $\mathbb{P}(X_1 = 1) = p$, $\mathbb{P}(X_1 = 0) = 1 - p$. Prove that for any $\varepsilon > 0$, $\mathbb{P}(S_n/n > p + \varepsilon) \leq e^{-\frac{1}{4} n \varepsilon^2}$.

**Exercise 7.** Let $X$ and $Y$ be two independent exponential random variables with parameter 1. What is the distribution of $X + Y$?

**Exercise 8.** Let $X$ and $Y$ be independent random variables uniform on $[0, 1]$. What is $\mathbb{E}(|X - Y|)$?

**Exercise 9.** Let the $X_\ell$’s be independent standard Cauchy random variables. Do $n^{-1} \sum_{\ell=1}^n X_\ell$ satisfy a law of large numbers? Do $n^{-1/2} \sum_{\ell=1}^n X_\ell$ satisfy a central limit theorem?

**Exercise 10.** Prove that if a sequence of real random variables $(X_n)$ converge in distribution to $X$, and $(Y_n)$ converges in distribution to a constant $c$, then $X_n + Y_n$ converges in distribution to $X + c$.

**Exercise 11.** Let the $X_\ell$’s be i.i.d. with mean 0 and variance $0 < \sigma^2 < \infty$. Does $n^{-\alpha} \sum_{\ell=1}^n X_\ell$ converge in distribution for $0 < \alpha < 1/2$? Same question for $\alpha = 1/2$ and $\alpha > 1/2$.

**Exercise 12.** Let $X_\ell$ and $Y_\ell$ be independent sequences of random variables, such that $X_\ell$ (resp. $Y_\ell$) converge in distribution to $X$ (resp. $Y$), with $X$ and $Y$ independent. Prove that $X_\ell + Y_\ell$ converges in distribution to $X + Y$. 

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Exercise 13. Prove that convergence in probability implies almost sure convergence along a subsequence.

Exercise 14 Prove that convergence in $L^p$ implies convergence in probability.

Exercise 15 Let the $X_\ell$’s be i.i.d. with a Gaussian distribution, with mean 2 and variance 2. What is the limit of $(X_1^2 + \cdots + X_n^2)/(X_1 + \cdots + X_n)$ as $n \to \infty$? In which sense?

Exercise 16 Let the $X_\ell$’s be independent random variables, and $0 < c_1 < c_2$ be absolute constants. Let $\mu_\ell = \mathbb{E}(X_\ell)$, and $\sigma_\ell^2 = \text{Var}(X_\ell)$ satisfy $c_1 < \sigma_\ell^2 < c_2$ for any $\ell$. State and prove a central limit theorem for $\sum_{\ell=1}^n X_\ell$. 