Exercise 1. Suppose that $\Omega$ is an infinite set (countable or not), and let $\mathcal{A}$ be the family of all subsets which are either finite or have finite complement. Prove that $\mathcal{A}$ is not a $\sigma$-algebra.

Exercise 2. Let $(A_n)_{n\geq 0}$ be a set of pairwise disjoint events and $\mathbb{P}$ a probability. Show that $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$.

Exercise 3. Prove the Bonferroni inequalities: if $A_i \in \mathcal{A}$ is a sequence of events, then
\[ \mathbb{P}(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j). \]

Exercise 4. A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first. Hint: consider the event $E_n$ that a 5 occurs on the $n$th roll and no 5 or 7 occurs on the first $(n-1)$ rolls.

Exercise 5. Let $\mathbb{P}$ be a probability measure on $\Omega$ endowed with a $\sigma$-algebra $\mathcal{A}$.
(i) What is the meaning of the following events, where all $A_n$’s are elements of $\mathcal{A}$?
\[ \liminf_{n \to \infty} A_n = \bigcup_{n \geq 1} \bigcap_{k \geq n} A_k, \quad \limsup_{n \to \infty} A_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k. \]
(ii) In the special case $\Omega = \mathbb{R}$ and $\mathcal{A}$ is its Borel $\sigma$-algebra, for any $p \geq 1$, let
\[ A_{2p} = \left[ -1, 2 + \frac{1}{2p} \right), \quad A_{2p+1} = \left( -2 - \frac{1}{2p+1}, 1 \right]. \]
What are $\liminf_{n \to \infty} A_n$ and $\limsup_{n \to \infty} A_n$?
(iii) Prove that the following always holds:
\[ \mathbb{P} \left( \liminf_{n \to \infty} A_n \right) \leq \liminf_{n \to \infty} \mathbb{P}(A_n), \quad \mathbb{P} \left( \limsup_{n \to \infty} A_n \right) \geq \limsup_{n \to \infty} \mathbb{P}(A_n). \]

Exercise 6. Let $n$ and $m$ be random numbers chosen independently and uniformly on $[1, N]$. What are $\Omega$, $\mathcal{A}$ and $\mathbb{P}$ (which all implicitly depend on $N$)? Prove that $\mathbb{P}(n \wedge m = 1) \underset{N \to \infty}{\longrightarrow} \zeta(2)^{-1}$ where $\zeta(2) = \prod_{p \in \mathcal{P}} (1 - p^{-2})^{-1} = \sum_{n \geq 1} n^{-2} = \frac{\pi^2}{6}$ (you don’t have to prove these equalities). Here $\mathcal{P}$ is the set of prime numbers and $n \wedge m = 1$ means that their greatest common divisor is 1.