Random Matrix Theory, homework 2, due April 7.

Problem 1. The Circular Unitary Ensemble is a log-correlated random field. Let \((e^{i\theta_k})_{1 \leq k \leq N}\) be the eigenvalues of a Haar-distributed matrix in \(U(N)\). The eigenangles have joint probability distribution

\[
\mathbb{P}(d\theta) = \frac{1}{N!} \prod_{1 \leq i < j \leq N} |e^{i\theta_i} - e^{i\theta_j}|^2 \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi}.
\]

(i) Prove that \(X = \sum_{k=1}^{N} \delta_{\theta_k}\) is a determinantal point process with correlation kernel

\[
K(x, y) = K^{(N)}(x, y) = \frac{1}{2\pi} \frac{\sin N\frac{x-y}{2}}{\sin \frac{x-y}{2}}
\]

with respect to the Lebesgue measure on \((0, 2\pi)\).

(ii) Let \(\phi : [0, 2\pi) \rightarrow \mathbb{R}\) be bounded measurable. Prove that

\[
\mathbb{E} \prod_{k=1}^{N} (1 + \phi(\theta_k)) = \sum_{n \geq 0} \frac{1}{n!} \int_{(0,2\pi)^n} \prod_{j=1}^{n} \phi(x_j) \det K(x_i, x_j) dx_1 \cdots dx_n.
\]

You will need to explain why the right hand side converges.

(iii) Read Section 3 in the book Trace ideals and applications.

(iv) Let \(A \subset [0, 2\pi)\) be measurable. On \(L^2(A)\), define \(K\phi\) the convolution operator with kernel \(K\phi\), where \(\phi\) is bounded measurable:

\[
(K\phi)(f)(x) = \int K(x, y)\phi(y) f(y) dy.
\]

Prove that \(K 1_A\) is trace-class with spectrum in \([0, 1]\). Let \(X = \chi(A)\). Show that

\[
\log \mathbb{E}(e^{\xi X}) = \log \det (\text{Id} + K 1_A(e^{\xi} - 1)) = -\sum_{k=1}^{\infty} \frac{1 - e^\xi)^k}{k} \text{Tr}((K 1_A)^k).
\]

(v) The formula \(\log \mathbb{E}(e^{\xi X}) = \sum_{\ell=1}^{\infty} C_\ell(X) \frac{\xi^\ell}{\ell!}\) defines the cumulants \(C_\ell(X)\) of the random variable \(X\). Prove that for any \(\ell \geq 3\),

\[
C_\ell(X) = (-1)^\ell (\ell - 1)! \text{Tr}(K 1_A - (K 1_A)^\ell) + \sum_{j=2}^{\ell-1} \alpha_j \ell C_j(X)
\]

for some universal constants \(\alpha_j\).

(vi) Take \(A = [0, x)\) \((x \in (0, 2\pi))\) in this question and the next one. Prove that

\[
C_2(X) = \int_0^x du \int_u^2 du \ |K(u, v)|^2 \sim_{N \to \infty} \pi^{-2} \log N.
\]

(vii) Prove that \(C_\ell(X / \sqrt{\log N})\) converges to 0 as \(N \to \infty\) for any \(\ell \geq 3\). For this you can first prove the trace inequality

\[
0 \leq \text{Tr}(K 1_A - (K 1_A)^\ell) \leq (\ell - 1)! \text{Tr}(K 1_A - (K 1_A)^\ell).
\]

Show that \((X - \mathbb{E}X) / \sqrt{\log N}\) converges weakly to a Gaussian random variable with variance \(\pi^{-2}\). Compare this result to the case of \(N\) independent uniform points on the circle.

(viii) Consider \(X_k = \chi([0, x_k)) - N x_k / (2\pi)\) where \(x_k = N^{-\alpha_k}, 0 < \alpha_1 < \cdots < \alpha_\ell < 1\). Prove a joint central limit theorem for the random variables \(X_1, \ldots, X_\ell\) as \(N \to \infty\). Compare this result to the case of \(N\) independent uniform points on the circle.
Problem 2. Loop equations and linear statistics for the Gaussian Unitary Ensemble.

Consider the probability distribution of eigenvalues from the Gaussian Unitary Ensemble:

$$
\mu(d\lambda) = \frac{1}{Z_N} \prod_{1 \leq k < \ell \leq N} |\lambda_k - \lambda_\ell|^2 e^{-\frac{N}{2} \sum_{k=1}^{N} \lambda_k^2} d\lambda_1 \ldots d\lambda_N
$$
on the simplex $\lambda_1 < \cdots < \lambda_N$. For a smooth function $f : \mathbb{R} \to \mathbb{R}$ supported on $(-2+\kappa, 2-\kappa)$ ($\kappa > 0$) we consider the general linear statistics $S_N(f) = \sum_{k=1}^{N} f(\lambda_k) - N \int f(s) \varrho(s) ds$, where $\varrho(s) = (2\pi)^{-1} \sqrt{(4 - s^2)}$. We want to prove the weak convergence of $S_N(f)$ to a Gaussian random variable for large $N$, with no need of any normalization.

We are interested in the Fourier transform $Z(u) = \mathbb{E}_\mu(e^{iuS_N(f)})$. We will need a complex modification of the GUE, namely $d\mu^u(\lambda) = \frac{e^{iu\lambda}}{Z(u)} d\mu(\lambda)$, assuming that $Z(u) \neq 0$. Let $s_N(z) = \frac{1}{N} \sum_{s} \frac{1}{z - \lambda_s}$ and $m_{N,u}(z) = \mathbb{E}^u(s_N(z))$. The Stieltjes transform of the semicircle distribution is $m(z) = \int \frac{\varrho(s)}{z - s} ds = \frac{z - \sqrt{z^2 - 4}}{2}$, where the square root is chosen so that $m$ is holomorphic on $[-2, 2]^c$ and $m(z) \to 0$ as $|z| \to \infty$.

(i) Prove that

$$
(m_{N,u}(z) - m(z))^2 - \sqrt{z^2 - 4} (m_{N,u}(z) - m(z)) + \frac{i}{N} \int_{\mathbb{R}} \frac{f'(s)}{z - s} \varrho_1^{(N,u)}(s) ds = -\text{var}_{\mu^u}(s_N(z)).
$$

This is the called the (first) loop equation. To derive it, you may first prove that

$$
m_{N,u}(z)^2 + \int_{\mathbb{R}} \frac{-s + iuN^{-1} f'(s)}{z - s} \varrho_1^{(N,u)}(s) ds = -\text{var}_{\mu^u}(s_N(z)).
$$

Hint: integrate by parts or change variables $\lambda_s = y_k + \epsilon(\Re e^{i s \lambda_k}) \frac{1}{z - y_k}$ and note $\partial_s \log Z(u) = 0$.

(ii) Remember the rigidity for Wigner matrices, in particular for GUE: for any $\xi, D > 0$ there exists $C > 0$ such that uniformly in $N \geq 1$ and $k \in [1, N]$ we have $\mu\left(|\lambda_k - \gamma_k| > N^{-\frac{4}{3}} + \xi \hat{\gamma}^{-\frac{4}{3}}\right) \leq C N^{-D}$, where $\int_{-\infty}^{\infty} \varrho(s) ds = \hat{\gamma}$ and $\hat{\gamma} = \min(k, N + 1 - k)$. Assume $Z(u) \neq 0$. Prove that

$$
|\mu^u|\left(|\lambda_k - \gamma_k| > N^{-\frac{4}{3}} + \xi \hat{\gamma}^{-\frac{4}{3}}\right) \leq C \frac{N^{-D}}{|Z(u)|},
$$

where $|\mu^u|$ is the total variation of the complex measure $\mu^u$. Conclude that uniformly in $z = E + i\eta$, $-2 + \kappa < E < 2 - \kappa$, $0 < |\eta| < 1$, we have

$$
|\text{var}_{\mu^u}(s_N(z))| = O\left(\frac{N^{-2+2\xi}}{|\eta|^2 |Z(u)|^2}\right).
$$

(iii) Prove that uniformly in $-2 + \kappa < E < 2 - \kappa$, $N^{-1+\xi} \leq \eta \leq 1$, we have

$$
m_{N,u}(z) - m(z) = \frac{1}{\sqrt{z^2 - 4}} \frac{i}{N} \int_{\mathbb{R}} \frac{f'(s)}{z - s} \varrho(s) ds + O\left(\frac{N^{-2+2\xi}}{|\eta|^2 |Z(u)|^2}\right).
$$

(iv) Let $\chi : \mathbb{R} \to \mathbb{R}^+$ be a smooth function such that $\chi(y) = 1$ for $|y| < 1/2$ and $\chi(y) = 0$ for $|y| > 1$. Prove that for any $\lambda \in \mathbb{R}$, we have

$$
f(\lambda) = -\frac{1}{2\pi i} \int_{\mathbb{R}^2} \frac{iyf''(x)\chi(y) + i(f(x) + iyf'(x))\chi'(y)}{x + iy - \lambda} dx dy,
$$

where the right hand side converges absolutely. For this, you can reproduce the proof of Cauchy's integral formula based on Green's theorem, considering the quasi-analytic extension $(f(x) + iyf'(x))\chi(y)$.

(v) Note that $\partial_u \log Z(u) = \mathbb{E}_{\mu^u}(iS_N(f))$. Conclude that bulk linear statistics converge to a Gaussian random variable.
Exercise 1. Fluctuations for the Ginibre ensemble. Consider the joint distribution of eigenvalues from the Ginibre ensemble,

\[ P(dz) = \frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 N^{1/2} \prod_{i=1}^{N} e^{-N|z_i|^2} dA(z_i) \]

where \( dA \) is the Lebesgue measure on \( \mathbb{C} \). Let \( \mathcal{C} \) be a smooth Jordan curve, with interior \( A \), finite length \( \ell(\mathcal{C}) \), strictly included in the unit disk \( \{ |z| < 1 \} \). Let \( X_{\mathcal{C}} = \chi(A) - E(\chi(A)) \) where \( \chi = \sum_{i=1}^{N} \delta_{z_i} \). By mimicking the method from Problem 1, prove the weak convergence

\[ \frac{X_{\mathcal{C}}}{\ell(\mathcal{C})^{1/2} N^{1/4}} \to N(0, c) \]

as \( N \to \infty \), with some \( c \) independent of \( \mathcal{C} \). What about joint convergence of \( (X_{\mathcal{C}_1}, \ldots, X_{\mathcal{C}_n}) \) where all Jordan curves \( \mathcal{C}_1, \ldots, \mathcal{C}_n \) satisfy the above assumptions?

Exercise 2. The semicircle law for band matrices. Let \( H_N \) be a symmetric matrix with \( H_N(i,j) \) a standard Bernoulli random variable when \( |i - j| \leq W/2 \) or \( |i - j| - N \leq W/2 \), 0 otherwise. All entries are independent, up to the symmetry constraint. Assume \( 1 \ll W \leq N \).

Prove that the empirical spectral measure of \( W^{-1/2} H_N \) converges (in probability, say) to the semi-circle distribution \( \varrho(s) = (2\pi)^{-1} \sqrt{(4 - s^2)} \).

Open problem 1. In Exercise 1, what happens when the Jordan curve is not smooth and has infinite length? In particular, if \( \log \text{var}(X_{\mathcal{C}}) \sim \alpha(\mathcal{C}) \log N \), does \( \alpha(\mathcal{C}) \) only depend on the Hausdorff dimension of \( \mathcal{C} \)? Or the Minkowski dimension?

Open problem 2. In Exercise 2, let \( u_1, \ldots, u_N \) be the \( L^2 \)-normalized eigenvectors of \( H_N \) and \( \alpha \in (0, 1), D > 0 \).

Assume \( \alpha < 1/2 \). Prove that there exists \( \delta > 0 \) such that for \( N \) greater than some \( N_0(\alpha, D) \), with probability at least \( 1 - N^{-D} \) the following holds: for any \( k \in [1, N] \), \( \| u_k \|_{\infty} > N^{-1/2+\delta} \).

Assume \( \alpha > 1/2 \). Prove that for any \( \delta > 0 \), for \( N \) greater than some \( N_0(\alpha, D, \delta) \), with probability at least \( 1 - N^{-D} \) the following holds: for any \( k \in [1, N] \), \( \| u_k \|_{\infty} < N^{-1/2+\delta} \).