Stochastic analysis, homework 2.

In all the following, $B$ is a Brownian motion.

Exercise 1
1 Calculate $\mathbb{E}(B_s^2 B_t^2)$, $\mathbb{E}(B_t \mid \mathcal{F}_s)$, $\mathbb{E}(B_t \mid B_s)$, for $t > s > 0$.
2 What is $\mathbb{E}(B_s^3 B_t^3)$, still for $t > s$?
3 What is the law of $B_t + B_s$? Same question for $\lambda_1 B_{t_1} + \cdots + \lambda_k B_{t_k}$ $(0 < t_1 < \cdots < t_k)$? What is the law of $\int_0^t B_s ds$?

Exercise 2 Convergence types.
1 Study the convergence in probability of $\lim t \to \infty \frac{\log(1+ B_t^2)}{\log t}$ as $t \to \infty$.
2 What about the almost sure convergence of $\lim t \to \infty \frac{\log(1+ B_t^2)}{\log t}$ as $t \to \infty$?

Exercise 3 Martingales from Brownian motion. Amongst the following processes, which ones are $\mathcal{F}$-martingales, where $\mathcal{F}$ is the natural filtration of $(B_s, s \geq 0)$? $B_t^2 - t$, $B_t^3 - 3 \int_0^t B_s ds$, $B_t^3 - 3t B_t$, $tB_t - \int_0^t B_s ds$.

Exercise 4 Scaling and equalities in law.
1 Let $T_a = \inf \{ t \mid B_t = a \}$ and $S_1 = \sup \{ B_s, s \leq 1 \}$. Prove that $T_a \overset{\text{law}}{=} a^2 T_1$.

Exercise 5 A convergence in law. Prove that as $t \to \infty$, $\left( \int_0^t e^{B_s} ds \right)^{1/\sqrt{t}}$ converges in law towards $e^{|\mathcal{N}|}$, where $\mathcal{N}$ is a standard Gaussian random variable.

Exercise 6 The zeros of Brownian motion. Let $Z = \{ t \geq 0 \mid B_t = 0 \}$.
1 Prove that $Z$ is almost surely a closed and unbounded set, with no isolated points.
2 Prove that $Z$ is almost surely uncountable.