Stochastic analysis, homework 3.

In all the following, $B$ is a Brownian motion, with completed right-continuous filtration $\mathcal{F}_t = \sigma(B_s, s \leq t)$.

**Exercise 1** Let $c$ and $d$ be two strictly positive numbers, $B$ a standard Brownian motion and $T = T_c \wedge T_d$.

1. Prove that, for every real number $s$,
   \[ \mathbb{E}\left(e^{-\frac{s^2}{2}T_{T_c}}\right) = \frac{\sinh(sd)}{\sinh(s(c + d))}. \]
   Prove that
   \[ \mathbb{E}\left(e^{-\frac{s^2}{2}T}\right) = \frac{\cosh(s(c - d)/2)}{\cosh(s(c + d)/2)}. \]

2. Prove that for $0 \leq s < \pi/(c + d)$,
   \[ \mathbb{E}\left(e^{\frac{s^2}{2}T}\right) = \frac{\cos(s(c - d)/2)}{\cos(s(c + d)/2)}. \]
   For this, use either a clearly justified analytic continuation or the complex martingale $e^{is(B_t - (c-d)/2 + s^2t/2)}$.

**Exercise 2** Let $f$ be a continuous function on $\mathbb{R}$.

1. Prove that if $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t \geq 0}$ martingale, then $f$ is affine.

2. Suppose that $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t \geq 0}$ submartingale: prove that $f$ has no proper local maximum. Hint: for $c > 0$, use the stopping times $T = T_c \wedge T_1$ and $S = \inf\{t \geq T : B_t = -1$ or $c + \epsilon$ or $c - \epsilon\}$.

3. Suppose that $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t \geq 0}$ submartingale: prove that $f$ is convex.

**Exercise 3** For $a > 0$, let $\sigma_a = \inf\{t \geq 0 : B_t < t - a\}$.

1. Prove that $\sigma_a$ is a.s. finite and that $\lim_{a \to \infty} \sigma_a = \infty$ almost surely.

2. Prove that $\mathbb{E}(e^{\frac{1}{2}\sigma_a}) = e^a$. For this, use the martingale $e^{-(\sqrt{1+2\lambda} - 1)(B_t - t) - \lambda t}$, and an analytic continuation.

3. Prove that the martingale $e^{B_t - \frac{1}{2}t}$ stopped at $\sigma_a$ is uniformly integrable.

4. For $a > 0$ and $b > 0$, define $\sigma_{a,b} = \inf\{t \geq 0 : B_t < bt - a\}$. Prove that $\sigma_{a,b} \overset{\text{law}}{=} b^{-2}\sigma_{ab,1}$ and
   \[ \mathbb{E}(e^{\frac{1}{2}b^2\sigma_{a,b}}) = e^{ab}. \]

5. For $b < 1$, prove that $\mathbb{E}(e^{\frac{1}{2}\sigma_{1,b}}) = \infty$. 

1