Stochastic analysis, homework 5.

Exercise 1 For a given Brownian motion $B$, let $X$ be a solution of
\[ dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x, \]
and $X^{(n)}$ be a solution of
\[ dX_t = \sigma^{(n)}(X_t)dB_t + b^{(n)}(X_t)dt, \quad X_0 = x, \]
where all functions are Lipschitz with the same absolute constant independent of $n$. Assume pointwise convergence of $\sigma^{(n)}$ to $\sigma$, and of $b^{(n)}$ to $b$. Prove that for any $t > 0$, as $n \to \infty$,
\[ \mathbb{E} \left( \sup_{[0,t]} |X_s - X_s^{(n)}|^2 \right) \to 0. \]

Exercise 2 Let $B$ be a Brownian motion, $a > 0$, $\gamma \geq 0$, and $T_{a,\gamma} = \inf \{ t \geq 0 \mid B_t + \gamma t = a \}$. Prove that the density of $T_{a,\gamma}$ with respect to the Lebesgue measure on $\mathbb{R}_+$ is
\[ \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{(a - \gamma t)^2}{2t}}. \]

Exercise 3 Let $B$ be a Brownian motion, $a > 0$, $\gamma \in \mathbb{R}$, and $S_{a,\gamma} = \inf \{ t \geq 0 \mid |B_t + \gamma t| = a \}$. Are $S_{a,\gamma}$ and $B_{S_{a,\gamma}} + \gamma S_{a,\gamma}$ independent under the Wiener measure?

Exercise 4 Let $X$ and $Y$ be independent Brownian motions.

1. Assume $X_0 = Y_0 = 0$, and note $T_a = \inf \{ t \geq 0 \mid X_t = a \}$ for $a > 0$. Prove that $T_a$ has the same law as $a^2/N^2$, where $N$ is a standard normal variable.

2. Prove that $Y_{T_a}$ has the same law as $aC$, where the Cauchy random variable $C$ is defined through its density with respect to the Lebesgue measure,
\[ \frac{1}{\pi(1+x^2)}. \]

3. Let $(X_0, Y_0) = (\epsilon, 0)$, where $0 < \epsilon < 1$. Note $Z_t = X_t + iY_t$. Justify that the winding number
\[ \theta_t = \frac{1}{2\pi} \arg Z_t \]
can be properly defined, continuously from $\theta_0 = 0$. Let $T^{(\epsilon)} = \inf \{ t \geq 0 \mid |Z_t| = 1 \}$. Prove that
\[ \frac{\theta_{T^{(\epsilon)}}}{\log \epsilon} \]
is distributed as $\frac{1}{2\pi} C$, $C$ being a Cauchy random variable.

4. Let $(X_0, Y_0) \neq (0, 0)$ and define as previously $Z_t = X_t + iY_t$ and $\arg Z_t$ continuously from $\arg Z_0 \in [0, 2\pi)$. Prove that, as $t \to \infty$,
\[ \frac{2 \arg Z_t}{\log t} \xrightarrow{\text{law}} C. \]