Stochastic calculus, homework 3, due October 3rd.

Exercise 1.
(i) Read carefully Theorems 1.9, 1.10 and 1.11 in the lecture notes.
(ii) Let $X_n, n \geq 0$, be i.i.d. real random variables such that $E(X_1) = 0$, $0 < E(|X_1|^2) < \infty$. For some parameter $\alpha > 0$, let
$$S_n = \sum_{k=1}^{n} \frac{X_k}{k^{\alpha}}.$$ Prove that if $\alpha > \frac{1}{2}$, $S_n$ converges almost surely. What if $0 < \alpha \leq \frac{1}{2}$?

Exercise 2. Let $B$ be a Brownian motion.
(i) Calculate $E(B_s B_t)$, $E(B_t \mid F_s)$, $E(B_t \mid B_s)$, for $t > s > 0$.
(ii) What is $E(B_s^2 B_t^2)$, still for $t > s$?
(iii) What is the law of $B_t + B_s$? Same question for $\lambda_1 B_{t_1} + \cdots + \lambda_k B_{t_k}$ ($0 < t_1 < \cdots < t_k$)? What is the law of $\int_0^1 B_s ds$?

Exercise 3. Let $B$ be a Brownian motion.
(i) Study the convergence in probability of $\log(1 + B_t^2) \log t$ as $t \to \infty$.
(ii) What about the almost sure convergence of $\log(1 + B_t^2) \log t$ as $t \to \infty$?

Exercise 4. Let $B$ be a Brownian motion, and for any $t \in [0,1]$ define
$$W_t = B_t - tB_1.$$ It is called a Brownian bridge.
(i) Prove that $W$ is a Gaussian process and calculate its covariance.
(ii) Let $0 < t_1 < \cdots < t_k < 1$. Prove that the vector $(W_{t_1}, W_{t_2}, \ldots, W_{t_k})$ has density
$$f(x_1, \ldots, x_k) = \sqrt{2\pi} p_{t_1}(x_1)p_{t_2-t_1}(x_2-x_1) \cdots p_{t_k-t_{k-1}}(x_k-x_{k-1})p_{1-t_k}(x_k)$$ where $p_t(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.
(iii) Prove that the law of $(W_{t_1}, W_{t_2}, \ldots, W_{t_k})$ is the same as the law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_k})$ conditionally to $B_1 = 0$.
(iv) Prove that the processes $(W_t)_{0 \leq t \leq 1}$ and $(W_{1-t})_{0 \leq t \leq 1}$ have the same distribution.

Exercise 5. Let $B$ be a Brownian motion, and for any $t \geq 0$ define
$$Z_t = B_t - \int_0^t \frac{B_s}{s} ds.$$ Prove that $Z$ is a Gaussian process and calculate its covariance. Does this process have a famous name?