Stochastic calculus, homework 3, due October 3rd.

Exercise 1.

(i) Read carefully Theorems 1.9, 1.10 and 1.11 in the lecture notes.

(ii) Let $X_n, n \geq 0$, be i.i.d. real random variables such that $\mathbb{E}(X_1) = 0, 0 < \mathbb{E}(|X_1|^2) < \infty$. For some parameter $\alpha > 0$, let

$$S_n = \sum_{k=1}^{n} \frac{X_k}{k^\alpha}.$$

Prove that if $\alpha > 1/2$, S_n converges almost surely. What if $0 < \alpha \leq 1/2$?

Exercise 2. Let B be a Brownian motion.

(i) Calculate $\mathbb{E}\left(B_s B_t^2\right)$, $\mathbb{E}\left(B_t \mid \mathcal{F}_s\right)$, $\mathbb{E}\left(B_t \mid B_s\right)$, for $t > s > 0$.

(ii) What is $\mathbb{E}\left(B_s^2 B_t^2\right)$, still for $t > s$?

(iii) What is the law of $B_t + B_s$? Same question for $\lambda_1 B_{t_1} + \cdots + \lambda_k B_{t_k}$ $(0 < t_1 < \cdots < t_k)$? What is the law of $\int_0^1 B_s \mathrm{d}s$?

Exercise 3. Let B be a Brownian motion.

(i) Study the convergence in probability of $\frac{\log(1+B_t^2)}{\log t}$ as $t \to \infty$.

(ii) What about the almost sure convergence of $\frac{\log(1+B_t^2)}{\log t}$ as $t \to \infty$?

Exercise 4. Let B be a Brownian motion, and for any $t \in [0,1]$ define

$$W_t = B_t - tB_1.$$

It is called a Brownian bridge.

(i) Prove that W is a Gaussian process and calculate its covariance.

(ii) Let $0 < t_1 < \cdots < t_k < 1$. Prove that the vector $(W_{t_1}, W_{t_2}, \dots, W_{t_k})$ has dentity

$$f(x_1, \dots, x_k) = \sqrt{2\pi} p_{t_1}(x_1) p_{t_2 - t_1}(x_2 - x_1) \dots p_{t_k - t_{k-1}}(x_k - x_{k-1}) p_{1-t_k}(x_k)$$

where $p_t(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

(iii) Prove that the law of $(W_{t_1}, W_{t_2}, \dots, W_{t_k})$ is the same as the law of $(B_{t_1}, B_{t_2}, \dots, B_{t_k})$ conditionally to $B_1 = 0$.

(iv) Prove that the processes $(W_t)_{0 \leq t \leq 1}$ and $(W_{1-t})_{0 \leq t \leq 1}$ have the same distribution.

Exercise 5. Let B be a Brownian motion, and for any $t \geq 0$ define

$$Z_t = B_t - \int_0^t \frac{B_s}{s} \mathrm{d}s.$$

Prove that Z is a Gaussian process and calculate its covariance. Does this process have a famous name?