Stochastic calculus, homework 5, due October 24th.

Exercise 1. Amongst the following processes, which ones are \( \mathcal{F} \)-martingales, where \( \mathcal{F} \) is the natural filtration of \((B_s, s \geq 0)\) (i.e. \( \mathcal{F}_t = \sigma(B_u, 0 \leq u \leq s) \))? \( B_t^2 - t, B_t^3 - 3 \int_0^t B_s ds, B_t^3 - 3tB_t, tB_t - \int_0^t B_s ds. \)

Exercise 2. Let \( B \) and \( \tilde{B} \) be two independent standard Brownian motions, and \( \rho \in [0, 1] \). Prove that \( \rho B + \sqrt{1 - \rho^2} \tilde{B} \) is a standard Brownian motion.

Let \( B = (B^{(1)}, \ldots, B^{(d)})' \) be a column vector with independent standard Brownian motions as entries. Let \( U \) be an orthogonal matrix. Show that the entries of \( UB \) are also independent Brownian motions.

Exercise 3. Prove that
\[
E \left((S_t - K)_+\right) = xN(d_1(t)) - KN(d_2(t))
\]
where \( S_t = xe^{\sigma B_t - \frac{\sigma^2 t}{2}} \) and \( B \) is a standard Brownian motion. In the above formula, we used the notations \( d_1(t) = \frac{1}{\sigma \sqrt{t}} \log \left( \frac{x}{K} + \frac{\sigma^2 t}{2} \right), d_2(t) = d_1(t) - \sigma \sqrt{t} \), and \( N(x) = \int_{-\infty}^{x} e^{-u^2/2} \sqrt{2 \pi} du \).

Exercise 4. Let \( c \) and \( d \) be two strictly positive numbers, \( B \) a standard Brownian motion and \( T = T_c \wedge T_{-d} \).

Prove the following Laplace transform identity: for every real number \( s \),
\[
E \left(e^{-s^2 t/2} \right) = \frac{\cosh(s(c - d)/2)}{\cosh(s(c + d)/2)}.
\]
By Taylor-expanding in \( s \), what are the expectation and variance of \( T \)?

Hint for the Laplace transform: follow the usual strategy applying a stopping time theorem to a pertinent martingale. This martingale is of exponential type.

Exercise 5: bonus. Prove that
\[
P \left( \sup_{s \leq u \leq t} B_u > 0, B_s < 0 \right) = 2P(B_t > 0, B_s < 0) = 2 \left( \frac{1}{4} - \frac{1}{2\pi} \arcsin \sqrt{\frac{s}{t}} \right).
\]
What is the distribution of \( g_t = \sup\{s \leq t : B_s = 0\} \)?