Exercise 1. Let $B$ be a Brownian motion. Prove that almost surely we have $\sup_{s\geq 0} B_s = +\infty$ and $\inf_{s\geq 0} B_s = -\infty$.

Exercise 2. Let $B$ be a Brownian motion and $T_a = \inf\{s \geq 0 : B_s = a\}$. Prove (with no calculation) that for any $b > a > 0$ the random variable $T_b - T_a$ is independent of $T_a$.

Simulate a Brownian motion (based on the method suggested by Donsker’s theorem, with a thousand time steps between times 0 and 1) and the corresponding process $(T_a)_{a \geq 0}$. Give a printed copy of the code with the homework (no matter which language) and a samples of both curves.

Exercise 3. Let $M$ and $N$ be two martingales bounded in $L^2$, and define

$$\langle M, N \rangle = \frac{1}{2}(\langle M + N \rangle - \langle N \rangle - \langle M \rangle).$$

Prove that $MN - \langle M, N \rangle$ is a martingale.

Exercise 4. Let $B$ be a Brownian motion starting at $x > 0$, and $T_0 = \inf\{s \geq 0 : B_s = 0\}$. That is the distribution of $\sup_{t \leq T_0} B_t$?

*Hint:* at some point in class we studied maxima of positive martingales converging to 0.

Exercise 5: bonus. Let $f$ be a continuous function on $\mathbb{R}$.

1. Prove that if $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t \geq 0}$ martingale, then $f$ is affine.

2. Suppose that $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t \geq 0}$ submartingale: prove that $f$ has no proper local maximum. Hint: for $c > 0$, use the stopping times $T = T_c \wedge T_{-1}$ and $S = \inf\{t \geq T : B_t = -1 \text{ or } c + \epsilon \text{ or } c - \epsilon\}$.

3. Suppose that $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t \geq 0}$ submartingale: prove that $f$ is convex.