Stochastic calculus, midterm exam

Lecture notes are not allowed. Six exercises perfectly solved give the maximum grade 100/100.

Exercise 1. Let \((X_i)_{i \geq 1}\) be i.i.d. uniform on \([0, 1]\). What is the limit of 
\[
\frac{X_1 + \cdots + X_n}{X_1^2 + \cdots + X_n^2}
\]
as \(n \to \infty\)?

Exercise 2. Let \(Y_1, Y_2, Y_3\) be independent \(\mathcal{N}(0, 1)\) random variables. Let 
\[
X_1 = Y_2 + 2Y_3, \quad X_2 = Y_1 - 3Y_2 + xY_3.
\]
for some real number \(x\).

(i) Explain why \(X = (X_1, X_2)\) is a Gaussian vector.
(ii) For what values of \(x\) are \(X_1\) and \(X_2\) independent?
(iii) Give an example of a vector with Gaussian entries which is not a Gaussian vector.

Exercise 3. Let \(X\) be a real Gaussian random variable with mean 0 and variance 7. Calculate 
(i) \(\mathbb{E}(e^{\lambda X})\) for any \(\lambda \in \mathbb{C}\);
(ii) \(\mathbb{E}(X^2 - 3X^2 + 12X - 4)\).

Exercise 4. Let \(X_1, X_2, \ldots\) be independent random variables and \(\mathbb{P}(X_j = 1) = \mathbb{P}(X_j = -1) = \mathbb{P}(X_j = 0) = 1/3\) for any \(j\). Let 
\(S_n = X_1 + \cdots + X_n\) and \(\mathcal{F}_n = \sigma(X_1, \ldots, X_n)\).

(i) Calculate \(\mathbb{E}(S_n), \mathbb{E}(S_n^2)\).
(ii) Is \(S_n\) a \((\mathcal{F}_n)_{n \geq 0}\)-martingale?
(iii) If \(m < n\), calculate \(\mathbb{E}(S_n^2 | \mathcal{F}_m)\).
(iv) If \(m < n\), calculate \(\mathbb{E}(X_m | S_n)\).

Exercise 5. Let \(X_i, i \geq 1\), be i.i.d. random variables, \(X_i \geq 0, \mathbb{E}(X_i) = 1\). Prove that if \(Y_n = \prod_{k=1}^n X_k, \mathcal{F}_n = \sigma(X_k, k \leq n), (Y_n)_{n \geq 0}\) is a \((\mathcal{F}_n)\)-martingale.

Prove that if \(\mathbb{P}(X_1 = 1) < 1, Y_n\) converges to 0 almost surely.

Exercise 6. Let \((X_i)_{i \geq 1}\) be a sequence of i.i.d. random variables with mean 0 and finite variance \(\mathbb{E}(X_i^2) = \sigma^2 > 0\). Let 
\(S_n = X_1 + \cdots + X_n\).

(i) State Donsker’s theorem.
(ii) Prove that \(\lim_{n \to \infty} \mathbb{E}\left(\frac{|S_n|}{\sqrt{n}}\right) = \sqrt{\frac{2}{\pi}}\sigma\).

Exercise 7. Let \(B\) be a Brownian motion. Calculate \(\mathbb{E}\left(e^{\int_0^t B_s ds}\right)\).

Exercise 8. Let \(B\) be a Brownian motion and \(\mathcal{F}_t = \sigma(B_s, s \leq t)\).

(i) Show that \((B_t^2 - t)_{t \geq 0}\) is a \((\mathcal{F}_t)_{t \geq 0}\)-martingale.
(ii) Is there a deterministic function \((f_t)_{t \geq 0}\) such that \(e^{(B_t^2 - t) - f_t}_{t \geq 0}\) is a \((\mathcal{F}_t)_{t \geq 0}\)-martingale?

Exercise 9. State the stopping time theorem for uniformly integrable continuous martingales.
What are the main intermediate lemmas for the proof?

Exercise 10. Let \(X_n, n \geq 0\), be independent random variables. Assume that \(\mathbb{E}(X_j) = 0\) and there exists a \(\alpha > 0\) such that \(\mathbb{E}(|X_j|^2) = j^{-\alpha}\) for any \(j \geq 1\). Let 
\(S_n = \sum_{k=1}^n X_k\). For which values of \(\alpha\) does \(S\) converge almost surely?