Exercise 1. State the central limit theorem for partial sums from a sequence of i.i.d. Bernoulli random variables \((X_i)_{i \geq 1}\), where \(P(X_i = 1) = p, P(X_i = -1) = 1 - p\), \(p \in [0, 1]\).

Exercise 2. Let \((X_i)_{i \geq 1}\) be i.i.d. Gaussian with mean 1 and variance 3. Show that
\[
\lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{X_1^2 + \cdots + X_n^2} = \frac{1}{4} \text{ a.s.}
\]

Exercise 3. Find an example of real random variables \((X_n)_{n \geq 1}\), \(X\), in \(L^1\), such that \((X_n)_{n \geq 1}\) converges to \(X\) in distribution and \(E(X_n)\) converges, but not towards \(E(X)\).

Exercise 4. Let \((X_n)_{n \geq 1}\) be independent Gaussian such that \(E(X_i) = m_i, \text{var}(X_i) = \sigma_i^2, i \geq 1\). Let \(S_n = \sum_{i=1}^n X_i\) and \(F_n = \sigma(X_i, 1 \leq i \leq n)\).

a) Find sequences \((b_n)_{n \geq 1}\), \((c_n)_{n \geq 1}\) of real numbers such that \((S_n^2 + b_n b_n + c_n)_{n \geq 1}\) is a \((F_n)_{n \geq 1}\)-martingale.

b) Let \(\lambda \in \mathbb{R}\). Find a sequence \((a_n(\lambda))_{n \geq 1}\) such that \((e^{\lambda}S_n - a_n(\lambda))_{n \geq 1}\) is a \((F_n)_{n \geq 1}\)-martingale.

Exercise 5. The goal of this exercise is to justify simulation of Gaussian random variables from uniform ones.

Let \(U_1\) and \(U_2\) be two independent random variables, uniform on \([0, 1]\), \(\theta = 2\pi U_1\) and \(S = -\log U_2\).

i) Prove that \(S\) has an exponential distribution.

ii) Prove that \(R = \sqrt{2S}\) has density \(xe^{-x^2/2}\) on \(\mathbb{R}_+\). This is the Rayleigh distribution.

iii) Prove that \(X_1 = R \cos \theta\) and \(X_2 = R \sin \theta\) are independent Gaussian random variables. This is the Box-Muller method to simulate Gaussian random variables.