Exercise 1. Let \((X_n, n \geq 0)\) be a non-negative supermartingale. Show the following maximal inequality: for \(a > 0\),
\[
a P \left( \sup_{[0,n]} X_k > a \right) \leq E(X_0).
\]

Exercise 2. Let \(X_0 > 0\), and at time \(n + 1\) you get \(\epsilon_n Y_n\) where \(Y_n\) was your stake at time \(n\), the \(\epsilon_n\)'s are iid and \(P(\epsilon_n = 1) = p = 1 - P(\epsilon_n = -1)\), \(p \in (1/2, 1)\): what you own at time \(n + 1\) is
\[
X_{n+1} = X_n + \epsilon_{n+1} Y_n,
\]
where \(Y_n \in \mathcal{F}_n\), \(0 \leq Y_n \leq X_n\), \(\mathcal{F}_n = \sigma(\epsilon_1, \ldots, \epsilon_n)\).

You want to maximize the expected return \(E \left( \log \frac{X_n}{X_0} \right)\), by finding the good strategy, i.e. what suitable \(\mathcal{F}_n\)-measurable function \(Y_n\) to choose. Prove that for some \(\lambda \geq 0\) explicit in terms of \(p\), \(((\log X_n) - n\lambda, n \geq 0)\) is a \((\mathcal{F}_n)\)-supermartingale, so that
\[
E \left( \log \frac{X_n}{X_0} \right) \leq n\lambda.
\]
Find a strategy such that equality occurs in the above equation.

Exercise 3. Consider the random walk \(S_n = \sum_1^n X_k\), \(S_0 = 0\), the \(X_k\)'s being iid, \(P(X_1 = 1) = P(X_1 = -1) = 1/2\), \(\mathcal{F}_n = \sigma(X_i, 0 \leq i \leq n)\).

Prove that \((S_n^2 - n, n \geq 0)\) is a \((\mathcal{F}_n)\)-martingale. Let \(\tau\) be a bounded stopping time. Prove that \(E(S_\tau^2) = \tau\).

Take now \(\tau = \inf\{n \mid S_n \in \{-a, b\}\}\), where \(a, b \in \mathbb{N}^*\). Prove that \(E(S_\tau) = 0\) and \(E(S_\tau^2) = E(\tau)\). What is \(P(S_\tau = -a)\)? What is \(E(\tau)\)?

By justifying the limit \(b \to \infty\), prove that the expectation of the hitting time of \(-a\) is infinity.

Exercise 4. In a game between a gambler and a croupier, suppose that the total capital in play is 1. After the \(n\)th hand the proportion of the capital held by the gambler is denoted \(X_n \in [0, 1]\), thus that held by the croupier is \(1 - X_n\). We assume \(X_0 = p \in (0, 1)\). The rules of the game are such that after \(n\) hands, the probability for the gambler to win the \((n + 1)\)th hand is \(X_n\); if he does, he gains half of the capital the croupier held after the \(n\)th hand, while if he loses he gives half of his capital. Let \(\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)\).

a) Show that \((X_n)_{n \geq 0}\) is a \((\mathcal{F}_n)_{n \geq 0}\) martingale.
b) Show that \((X_n)_{n \geq 1}\) converges a.s. and in \(L^2\) towards a limit \(Z\).
c) Show that \(E(X_{n+1}^2) = E(3X_n^2 + X_n)/4\). Deduce that \(E(Z^2) = E(Z) = p\). What is the law of \(Z\)?
d) For any \(n \geq 0\), let \(Y_n = 2X_{n+1} - X_n\). Find the conditional law of \(X_{n+1}\) knowing \(\mathcal{F}_n\). Prove that \(P(Y_n = 0 \mid \mathcal{F}_n) = 1 - X_n\), \(P(Y_n = 1 \mid \mathcal{F}_n) = X_n\) and express the law of \(Y_n\).
e) Let $G_n = \{Y_n = 1\}, L_n = \{Y_n = 0\}$. Prove that $Y_n \to Z$ a.s. and deduce that $\mathbb{P}(\lim \inf_{n \to \infty} G_n) = p$, $\mathbb{P}(\lim \inf_{n \to \infty} L_n) = 1 - p$. Are the variables $\{Y_n, n \geq 1\}$ independent?

f) Interpret the questions c), d), e) in terms of gain, loss, for the gambler.

**Exercise 5 (bonus).** Let $X$ be a standard random walk in dimension 1, and for any positive integer $a$, $\tau_a = \inf \{n \geq 0 \mid X_n = a\}$. For any $\theta > 0$, calculate $\mathbb{E} \left( (\cosh \theta)^{-\tau_a} \right)$.

Hint: look for a pertinent martingale of exponential type.