THE CHARACTERISTIC POLYNOMIAL OF A RANDOM UNITARY MATRIX: A PROBABILISTIC APPROACH

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Abstract. In this paper, we propose a probabilistic approach to the study of the characteristic polynomial of a random unitary matrix. We recover the Mellin Fourier transform of such a random polynomial, first obtained by Keating and Snaith in [7], using a simple recursion formula, and from there we are able to obtain the joint law of its radial and angular parts in the complex plane. In particular, we show that the real and imaginary parts of the logarithm of the characteristic polynomial of a random unitary matrix can be represented in law as the sum of independent random variables. From such representations, the celebrated limit theorem obtained by Keating and Snaith in [7] is now obtained from the classical central limit theorems of Probability Theory, as well as some new estimates for the rate of convergence and law of the iterated logarithm type results.

1. Introduction

In [7], Keating and Snaith argued that the Riemann zeta function on the critical line could be modelled by the characteristic polynomial of a random unitary matrix considered on the unit circle. In their development of the model they showed, via calculating the Mellin-Fourier transform, that the logarithm of the characteristic polynomial weakly converges to a normal distribution, analogous to Selberg's result on the normal distribution of values of the logarithm of the Riemann zeta function [17].

The purpose of this paper is to prove an equality in law between the characteristic polynomial and products of independent random variables. Using this we rederive the limit theorem and Mellin-Fourier transform of Keating and Snaith and prove some new results about the speed of convergence.

Let V_N denote a generic $N \times N$ random matrix drawn from the unitary group $U(N)$ with the Haar measure $\mu_{U(N)}$. The characteristic polynomial of V_N is

$$\begin{aligned} Z(V_N, \theta) &:= \det(I_N - e^{-\mathrm{i}\theta} V_N) \\ &= \prod_{j=1}^{N} \left(1 - e^{\mathrm{i}(\theta_n - \theta)}\right) \end{aligned}$$

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where $e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_N}$ are the eigenvalues of V_N. Note that by the rotation invariance of Haar measure, if θ is real then $Z(V_N,\theta)\overset{\text{law}}{=}Z(V_N,0)$. Therefore here and in the following we may simply write Z_N for $Z(V_N,\theta)$. Keating and Snaith [7] evaluated the Mellin-Fourier transform of Z_N. Integrating against the Weyl density for Haar measure on $U(N)$, and using certain Selberg integrals, they obtained, for all t and s with $\mathfrak{Re}(t\pm s)>-1$,

$$\mathbb{E}\left(|Z_N|^t e^{\mathrm{i}s\arg Z_N}\right)=\prod_{k=1}^{N}\frac{\Gamma(k)\,\Gamma(k+t)}{\Gamma\left(k+\frac{t+s}{2}\right)\Gamma\left(k+\frac{t-s}{2}\right)}. \tag{1.1}$$

In [7] and in this article, $\arg Z_N$ is defined as the imaginary part of

$$\log Z_N:=\sum_{n=1}^{N}\log(1-e^{\mathrm{i}\theta_n})$$

with $\mathfrak{Im}\log(1-e^{\mathrm{i}\theta_n})\in(-\pi/2,\pi/2]$. An equivalent definition for $\log Z_N$ is the value at point $x=1$ of the unique continuous function $\log\det(I_N-xV_N)$ (on $[0,1]$) which is 0 at $x=0$.

By calculating the asymptotics of the cumulants of (1.1), they were able to show that for any fixed s,t,

$$\mathbb{E}\left(|Z_N|^{t/\sqrt{(\log N)/2}}e^{\mathrm{i}s\arg Z_N/\sqrt{(\log N)/2}}\right)\to\exp\left(\tfrac{1}{2}t^2-\tfrac{1}{2}s^2\right)$$

as $N\to\infty$, and from this deduce the central limit theorem

$$\frac{\log Z_N}{\sqrt{\frac{1}{2}\log N}}\xrightarrow{\text{law}}\mathcal{N}_1+\mathrm{i}\mathcal{N}_2,$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are two independent standard Gaussian random variables.

We will see in the two following sections how (1.1) may be simply interpreted as an identity in law involving a certain product of independent random variables. In particular, we shall show that $\mathfrak{Re}\log Z_N$ and $\mathfrak{Im}\log Z_N$ can be written in law as sums of independent random variables. Sums of independent random variables are very well known and well studied objects in Probability Theory, and we can thus have a better understanding of the distribution of the characteristic polynomial with such a representation. We also apply the classical limit theorems on such sums to obtain asymptotic properties of Z_N when $N\to\infty$. In particular, we recover the convergence in law of $\log Z_N/\sqrt{\frac{1}{2}\log N}$ to a standard complex Gaussian law as a consequence of the classical central limit theorem. We also obtain some new results about the rate of convergence and prove an iterated logarithm law for the characteristic polynomial.

More precisely, the paper is organized as follows: in Section 2, we use a recursive construction for the Haar measure on $U(N)$, to obtain our first equality in law for the distribution of the characteristic polynomial as a product of independent random variables, from which we obtain a new proof of

(1.1) which does not use Selberg's integrals or the Weyl density. Then in Section 3 we use (1.1) to deduce the joint law of $(\mathfrak{Re}\log Z_N, \mathfrak{Im}\log Z_N)$, writing each component as a sums of independent random variables. Using these two representations for Z_N, in Section 4 two new proofs of Keating-Snaith limit theorem (the convergence in law of $\log Z_N/\sqrt{\frac{1}{2}\log N}$ to a standard complex Gaussian law) are provided. We also give estimates on the rate of convergence in the central limit theorem. In Section 5 we see how strong limit theorems such as the iterated logarithm can be deduced from our representations.

In a companion paper to [7], Keating and Snaith [8] studied the characteristic polynomial for classical compact groups other than the unitary group, and in Section 6, we also give similar results for $SO(2N)$ which plays a similar role to $U(N)$ for other families of L-functions.

Since the publication of [7], there have been many developments in understanding the distribution of the characteristic polynomial. For example, other limit theorems and large deviation results were derived for the characteristic polynomial by Hughes, Keating and O'Connell in [5]. The distribution of the characteristic polynomial away from the point $\theta = 0$ in the other groups has been studied by Odgers [12]. For more details about the connections between random matrix theory and analytic number theory, see [10] and the references therein, or the excellent survey article by Royer [16].

2. Decomposition of the Haar measure

In this section, we give an alternative proof of formula (1.1), which does not necessitate the explicit knowledge of the Weyl density formula and the values of some Selberg integrals. This new demonstration relies on a recursive presentation of the Haar measure $\mu_{U(N)}$.

2.1. **Decomposition of the Haar measure.** Let V_N be distributed with Haar measure $\mu_{U(N)}$ on $U(N)$. If $M \in U(N+1)$ is independent of V_N a natural question to ask is under which condition on the distribution of M, is the matrix

$$M\begin{pmatrix} 1 & 0 \\ 0 & V_N \end{pmatrix} \tag{2.1}$$

distributed with the Haar measure in dimension $N+1$?

The solution to such a question allows one to recursively build up a Haar distributed element of $U(N)$.

This question was partially answered by Mezzadri relying on a general method due to Diaconis and Shahshahani [4]. Due to a factorization of the Ginibre ensemble, Mezzadri showed that when M is a suitable Householder reflection then (2.1) is distributed with the Haar measure in dimension $N+1$. More precisely, he showed that if v is a unit vector chosen uniformly on the $(N+1)$-dimensional unit complex-sphere

$$\mathscr{S}_{\mathbb{C}}^{N+1} := \{(c_1, \ldots, c_{N+1}) \in \mathbb{C}^{N+1} \ : \ |c_1|^2 + \cdots + |c_{N+1}|^2 = 1\},$$

and if θ is the argument of the first coordinate of v, and u is the unit vector along the bisector of e_1 and v, where $e_1 = (1, 0, \dots, 0)$ is the unit vector for the first coordinate, then one could take M to be an element of $U(N+1)$ that can be written $-e^{-\mathrm{i}\theta}(I_{N+1} - 2u\overline{u}^T)$.

By the application $M \mapsto M_1$, it is clear that a necessary condition for our question must be that v must be distributed according to the uniform measure on $\mathscr{S}_{\mathbb{C}}^{N+1}$. The following proposition states that this condition is also sufficient. It is a slight generalization of Mezzadri's result, and its proof does not require a decomposition of the Ginibre ensemble.

Proposition 2.1. *Let $M \in U(N+1)$ be such that its first column M_1 is uniformly distributed on $\mathscr{S}_{\mathbb{C}}^{N+1}$. If $V_N \in U(N)$ is chosen independently of M according to the Haar measure $\mu_{U(N)}$, then the matrix*

$$V_{N+1} := M \begin{pmatrix} 1 & 0 \\ 0 & V_N \end{pmatrix}$$

is distributed with the Haar measure $\mu_{U(N+1)}$.

Proof. Due to the uniqueness property of the Haar measure, we only need to show that for a fixed $U \in U(N+1)$

$$UM \begin{pmatrix} 1 & 0 \\ 0 & V_N \end{pmatrix} \overset{\text{law}}{=} M \begin{pmatrix} 1 & 0 \\ 0 & V_N \end{pmatrix}.$$

In the following, a matrix A will often be written $(A_1 \| \tilde{A})$, where A_1 is its first column. As $U \in U(N+1)$, $(UM)_1 = UM_1$ is distributed uniformly on the complex unit sphere $\mathscr{S}_{\mathbb{C}}^{N+1}$, so we can write $UM = (P_1 \| \tilde{P})$, with P_1 uniformly distributed on $\mathscr{S}_{\mathbb{C}}^{N+1}$ and $\tilde{P}$ having a distribution on the orthogonal hyperplane of P_1. We then need to show that

$$(P_1 \| \tilde{P}) \begin{pmatrix} 1 & 0 \\ 0 & V_N \end{pmatrix} \overset{\text{law}}{=} (M_1 \| \tilde{M}) \begin{pmatrix} 1 & 0 \\ 0 & V_N \end{pmatrix},$$

where all matrices are still independent. As M_1 and P_1 are identically distributed, by conditioning on $M_1 = P_1 = v$ (here v is any fixed element of $\mathscr{S}_{\mathbb{C}}^{N+1}$) it is sufficient to show that

$$(v \| P') \begin{pmatrix} 1 & 0 \\ 0 & V_N \end{pmatrix} \overset{\text{law}}{=} (v \| M') \begin{pmatrix} 1 & 0 \\ 0 & V_N \end{pmatrix},$$

where M' (resp P') is distributed like $\tilde{M}$ (resp $\tilde{P}$) conditionally to $M_1 = v$ (resp $P_1 = v$). Let A be any element of $U(N+1)$ such that $A(v) = (1, 0, \dots, 0)$. Since A is invertible, we just need to show that

$$A(v \| P') \begin{pmatrix} 1 & 0 \\ 0 & V_N \end{pmatrix} \overset{\text{law}}{=} A(v \| M') \begin{pmatrix} 1 & 0 \\ 0 & V_N \end{pmatrix},$$

that is to say

$$P'' V_N \overset{\text{law}}{=} M'' V_N,$$

where P'' and M'' are distributed on $U(N)$ independently of V_N. By independence and conditioning on P'' (resp M''), we get $P''V_N \overset{\text{law}}{=} V_N$ (resp $M''V_N \overset{\text{law}}{=} V_N$) by definition of the Haar measure $\mu_{U(N)}$. This gives the desired result. □

The result of this proposition is very natural. It states, roughly speaking, that in order to choose uniformly an element of $U(N+1)$ (that is to say an orthogonal unitary basis) one just needs to choose the first element uniformly on the sphere and then an element of $U(N)$ in the hyperplane orthogonal to the first element, uniformly.

2.2. **Recovering the Mellin Fourier transform.** The decomposition of the Haar measure presented in the previous paragraph gives another proof for equation (1.1). In reality, the following Proposition 2.2 gives much more, as we get a representation of $Z(V_N)$ as a product of N simple independent random variables.

Proposition 2.2. *Let $V_N \in U(N)$ be distributed with the Haar measure $\mu_{U(N)}$. Then for all $\theta \in \mathbb{R}$*

$$\det(I_N - e^{i\theta}V_N) \overset{\text{law}}{=} \prod_{k=1}^{N} \left(1 + e^{\mathrm{i}\theta_k}\sqrt{\beta_{1,k-1}}\right),$$

with $\theta_1, \dots, \theta_n, \beta_{1,0}, \dots, \beta_{1,n-1}$ independent random variables, the θ_k's uniformly distributed on $[0, 2\pi]$ and the $\beta_{1,j}$'s ($0 \le j \le N-1$) being beta distributed with parameters 1 and j (by convention, $\beta_{1,0}$ is the Dirac distribution on 1).

Proof. As previously mentioned, it suffices to consider the case $\theta = 0$.

Note now that in Proposition 2.1, we can choose any matrix $M \in U(N)$ with M_1 uniformly distributed on the complex sphere $\mathscr{S}_{\mathbb{C}}^N$. Let us choose the simplest suitable transformation M : the reflection with respect to the median hyperplane of e_1 and M_1, where M_1 is chosen uniformly on $\mathscr{S}_{\mathbb{C}}^N$. Let the vector v be $M_1 - e_1$. Therefore there exists $(\lambda_2, \dots, \lambda_N) \in \mathbb{C}^{N-1}$ such that

$$M = (e_1 + v \| e_2 + \lambda_2 v \| \dots \| e_N + \lambda_N v).$$

So, with Proposition 2.1, one can write

$$\begin{aligned}\det(I_N - V_N) &\overset{\text{law}}{=} \det\left[I_N - M\begin{pmatrix} 1 & 0 \\ 0 & V_{N-1}\end{pmatrix}\right] \\ &= \det\left[\begin{pmatrix} 1 & 0 \\ 0 & \overline{V}_{N-1}^T\end{pmatrix} - M\right]\det\begin{pmatrix} 1 & 0 \\ 0 & V_{N-1}\end{pmatrix}.\end{aligned}$$

If we call $(u_1\|\dots\|u_{N-1}) := \overline{V}_{N-1}^T$ then using the multi-linearity of the determinant we get

$$\begin{aligned}
&\det\left[\begin{pmatrix} 1 & 0 \\ 0 & \overline{V}_{N-1}^T \end{pmatrix} - M\right] \\
&\quad = \det\left(-v\left\| \begin{pmatrix} 0 \\ u_1 \end{pmatrix} - e_2 - \lambda_2 v \right\| \dots \left\| \begin{pmatrix} 0 \\ u_{N-1} \end{pmatrix} - e_N - \lambda_N v\right.\right) \\
&\quad = \det\left(-v\left\| \begin{pmatrix} 0 \\ u_1 \end{pmatrix} - e_2 \right\| \dots \left\| \begin{pmatrix} 0 \\ u_{N-1} \end{pmatrix} - e_N\right.\right) \\
&\quad = \det\left(\begin{array}{c|c} -v_1 & 0 \\ \dots & \overline{V}_{N-1}^T - I_{N-1} \end{array}\right) \\
&\quad = -v_1 \det\left(\overline{V}_{N-1}^T - I_{N-1}\right).
\end{aligned}$$

Finally,

$$\det(I_N - V_N) \overset{\text{law}}{=} -v_1 \det(I_{N-1} - V_{N-1}),$$

with $-v_1 = 1 - M_{11}$ and V_{N-1} independent. Therefore, to prove Proposition 2.2, we only need to show that $M_{11} \overset{\text{law}}{=} e^{\mathrm{i}\theta_N}\sqrt{\beta_{1,N-1}}$. This is straightforward because, since M_1 is a random vector chosen uniformly on $\mathscr{S}_{\mathbb{C}}^N$, we know that

$$M_{11} \overset{\text{law}}{=} \frac{x_1 + \mathrm{i}y_1}{\sqrt{(x_1^2 + y_1^2) + \dots + (x_N^2 + y_N^2)}} \overset{\text{law}}{=} e^{\mathrm{i}\theta_N}\sqrt{\beta_{1,N-1}},$$

with the x_i's and y_i's all independent standard normal variables, θ_N and $\beta_{1,N-1}$ as stated in Proposition 2.2. $\square$

To end the proof of (1.1), we now only need the following lemma.

Lemma 2.3. *Let $X := 1 + e^{\mathrm{i}\theta}\sqrt{\beta}$, where θ has uniform distribution on $[0, 2\pi]$ and, independently β has a beta law with parameters 1 and $N-1$. Then, for all t and s with $\mathfrak{Re}(t \pm s) > -1$*

$$\mathbb{E}\left(|X|^t e^{\mathrm{i}s \arg X}\right) = \frac{\Gamma(N)\,\Gamma(N+t)}{\Gamma\left(N + \frac{t+s}{2}\right)\Gamma\left(N + \frac{t-s}{2}\right)}.$$

Proof. First, note that

$$\begin{aligned}
\mathbb{E}\left(|X|^t e^{\mathrm{i}s \arg X}\right) &= \mathbb{E}\left(X^{(t+s)/2}\overline{X}^{(t-s)/2}\right) \\
&= \mathbb{E}\left(\left(1 + e^{\mathrm{i}\theta}\sqrt{\beta}\right)^a \left(1 + e^{-\mathrm{i}\theta}\sqrt{\beta}\right)^b\right),
\end{aligned}$$

with $a = (t+s)/2$ and $b = (t-s)/2$. Recall that if $|x| < 1$ and $u \in \mathbb{R}$ then

$$(1+x)^u = \sum_{k=0}^{\infty} \frac{u(u-1)\dots(u-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \frac{(-1)^k(-u)_k}{k!} x^k,$$

where $(y)_k = y(y+1)\dots(y+k-1)$ is the Pochhammer symbol. As $|e^{\mathrm{i}\theta}\sqrt{\beta}| < 1$ a.s., we get

$$\begin{aligned}\mathbb{E}\left[|X|^t e^{\mathrm{i}s\arg X}\right] \\ &= \mathbb{E}\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k(-a)_k}{k!}\beta^{\frac{k}{2}}e^{\mathrm{i}k\theta}\right)\left(\sum_{\ell=0}^{\infty}\frac{(-1)^\ell(-b)_\ell}{\ell!}\beta^{\frac{\ell}{2}}e^{-\mathrm{i}\ell\theta}\right)\right].\end{aligned}$$

After an expansion of this double sum (it is absolutely convergent because $\mathfrak{Re}(t \pm s) > -1$), all terms with $k \neq \ell$ will give an expectation equal to 0, because we integrate with respect to the uniform probability measure along the unit circle. So we get

$$\mathbb{E}\left[|X|^t e^{\mathrm{i}s\arg X}\right] = \mathbb{E}\left[\sum_{k=0}^{\infty}\frac{(-a)_k(-b)_k}{(k!)^2}\beta^k\right].$$

As β is a beta variable with parameters 1 and $N-1$, we have

$$\mathbb{E}\left[\beta^k\right] = \frac{\Gamma(1+k)\Gamma(N)}{\Gamma(1)\Gamma(N+k)} = \frac{k!}{(N)_k},$$

hence

$$\mathbb{E}\left[|X|^t e^{\mathrm{i}s\arg X}\right] = \sum_{k=0}^{\infty}\frac{(-a)_k(-b)_k}{k!(N)_k}.$$

Note that this series is equal to the value at $z = 1$ of the hypergeometric function $H(-a,-b,N;z)$. This value is well known (see, for example, [1]) and yields:

$$\mathbb{E}\left[|X|^t e^{\mathrm{i}s\arg X}\right] = \frac{\Gamma(N)\Gamma(N+a+b)}{\Gamma(N+a)\Gamma(N+b)}.$$

This gives the desired result. □

Comments about Selberg integrals. To prove (1.1) Keating and Snaith [7], relying on Weyl's integration formula, used the result by Selberg

$$\begin{aligned}J(a,b,\alpha,\beta,\gamma,N) &:= \\ &\int\cdots\int_{\mathbb{R}^N}\prod_{1\le j<\ell\le N}|x_j-x_\ell|^{2\gamma}\prod_{j=1}^{N}(a+ix_j)^{-\alpha}(b-ix_j)^{-\beta}\mathrm{d}x_j \\ &= \frac{(2\pi)^N}{(a+b)^{(\alpha+\beta)N-\gamma N(N-1)-N}}\prod_{j=0}^{N-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+\beta(N+j-1)\gamma-1)}{\Gamma(1+\gamma)\Gamma(\alpha-j\gamma)\Gamma(\beta-j\gamma)},\end{aligned} \tag{2.2}$$

in the specific case $a = b = 1$ and $\gamma = 1$. Thus our probabilistic proof for (1.1) also gives an alternative proof of (2.2) for these specific values of the parameters. Moreover, as we will see in Section 6, a similar result as Proposition 2.2 still holds for the orthogonal group. As a consequence Selberg's integral formula admits a probabilistic proof for $a = b$ and $\gamma = 1/2$

and 1 (however this method relies on Weyl's integration formula, which is essentially analytical).

2.3. **Decomposition of the characteristic polynomial off the unit circle.** Proposition 2.2 can be extended to the law of the characteristic polynomial of a random unitary matrix off the unit circle where we replace $e^{i\theta}$ by a fixed x. Once more, due to the rotation invariance of the unitary group, we may take x to be real.

Proposition 2.4. *Let $x \in \mathbb{R}$, V_{N-1} distributed with the Haar measure $\mu_{U(N-1)}$, M_1 uniformly chosen on $\mathscr{S}_{\mathbb{C}}^N$, independently of U_{N-1}. We write M_{11} for the first coordinate of M_1, and $\tilde{M}_1$ for the vector with coordinates $M_{12}, \dots, M_{1N}$.*

Then, if V_N is distributed with the Haar measure $\mu_{U(N)}$,

$$\det(I_N - xV_N) \overset{\text{law}}{=} (1 - xM_{11})\ \det(I_{N-1} - xV_{N-1}) + \frac{x(1-x)}{1-\overline{M_{11}}}\overline{\tilde{M}_1}^T(\overline{V_{N-1}}^T - xI_{N-1})^{-1}\tilde{M}_1\ \det(I_{N-1} - xV_{N-1}). \quad (2.3)$$

Proof. The idea is the same as for Proposition 2.2, where we use Proposition 2.1 with a specific choice of M, the reflection with respect to the hyperplane median to $e_1 := (1, 0, \dots, 0)$ and M_1, a vector of $\mathscr{S}_{\mathbb{C}}^N$ chosen uniformly. If $k := M_1 - e_1$ we can write more precisely

$$M = \left(M_1, e_2 + \frac{-\overline{M}_{12}}{1-\overline{M}_{11}}k, \dots, e_N + \frac{-\overline{M}_{1N}}{1-\overline{M}_{11}}k\right).$$

Thus, using multi-linearity of the determinant, due to Proposition 2.1 we get after some straightforward calculation

$$\begin{aligned}\det(I_N - xV_N) &\overset{\text{law}}{=} \det\left[\begin{pmatrix} 1 & 0 \\ 0 & \overline{V}_{N-1}^T \end{pmatrix} - xM\right]\det\begin{pmatrix} 1 & 0 \\ 0 & V_{N-1}\end{pmatrix} \\ &= b\ \det\begin{pmatrix} a & \overline{\tilde{M}}_1^T \\ \tilde{M}_1 & \overline{V}_{N-1}^T - xI_{N-1}\end{pmatrix}\det\begin{pmatrix} 1 & 0 \\ 0 & V_{N-1}\end{pmatrix}\end{aligned}$$

with $b = \frac{-x(1-x)}{1-\overline{M}_{11}}$ and $a = \frac{(1-xM_{11})(1-\overline{M}_{11})}{-x(1-x)}$. As we want to express these terms with respect to $\det(I_{N-1} - V_{N-1})$, writing $B := \overline{V}_{N-1}^T - xI_{N-1}$ leads to

$$\begin{aligned}\det(I_N - xV_N) &\overset{\text{law}}{=} b\ \det\begin{pmatrix} a & \overline{\tilde{M}}_1^T \\ \tilde{M}_1 & B\end{pmatrix}\det\begin{pmatrix} 1 & 0 \\ -B^{-1}\tilde{M}_1 & V_{N-1}\end{pmatrix} \\ &= b\ \det\begin{pmatrix} a - \overline{\tilde{M}}_1^T B^{-1}\tilde{M}_1 & \cdots \\ 0 & BV_{N-1}\end{pmatrix} \\ &= b\ (a - \overline{\tilde{M}}_1^T B^{-1}\tilde{M}_1)\ \det(I_{N-1} - xV_{N-1}).\end{aligned}$$

This is the expected result. □

One may try to get a new proof of Weyl's integration formula thanks to this recursive construction of a characteristic polynomial. Let ν_N be the probability measure on $[0,2\pi)^N$ with density

$$\nu_N(\mathrm{d}a_1,\dots,\mathrm{d}a_N) = c_N \prod_{j\neq k} \left|e^{ia_j} - e^{ia_k}\right|^2 \mathrm{d}a_1\dots\mathrm{d}a_N.$$

It would be sufficient to show that if $(\theta_1,\dots,\theta_N)$ and $(\tilde\theta_1,\dots,\tilde\theta_{N-1})$ are independent and respectively distributed according to ν_N and ν_{N-1}, then for all $x\in\mathbb{R}$, with the notations of the proposition,

$$\prod_{j=1}^{N}(1-xe^{i\theta_j}) \overset{\text{law}}{=} (1-xM_{11})\prod_{j=1}^{N-1}(1-xe^{i\tilde\theta_j}) + \frac{x(1-x)}{1-\overline{M_{11}}}\sum_{j=1}^{N-1} e^{i\tilde\theta_j}|M_{1,j+1}|^2\prod_{k\neq j}\left(1-xe^{i\tilde\theta_k}\right).$$

However, this identity in law does not seem to have an easy direct explanation.

3. Decomposition into independent random variables

3.1. Some formulae about the beta-gamma algebra. We recall here some well known facts about the beta-gamma algebra which we shall often use in the sequel. A gamma random variable γ_a with coefficient $a>0$ has density given by:

$$\mathbb{P}\left\{\gamma_a\in\mathrm{d}t\right\} = \frac{t^{a-1}}{\Gamma(a)}e^{-t}\mathrm{d}t.$$

Its Mellin transform is $(s>0)$

$$\mathbb{E}\left[\gamma_a^s\right] = \frac{\Gamma(a+s)}{\Gamma(a)}.$$

A beta random variable $\beta_{a,b}$ with strictly positive coefficients a and b has density on $[0,1]$ given by

$$\mathbb{P}\left\{\beta_{a,b}\in\mathrm{d}t\right\} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}t^{a-1}(1-t)^{b-1}\mathrm{d}t.$$

Its Mellin transform is $(s>0)$

$$\mathbb{E}\left[\beta_{a,b}^s\right] = \frac{\Gamma(a+s)}{\Gamma(a)}\frac{\Gamma(a+b)}{\Gamma(a+b+s)}. \tag{3.1}$$

We will also make use of the following two properties (see [3] for justifications): the algebra property where all variables are independent

$$\beta_{a,b}\gamma_{a+b} \overset{\text{law}}{=} \gamma_a,$$

and the duplication formula for the gamma variables, with all variables independent

$$\gamma_j \overset{\text{law}}{=} 2\sqrt{\gamma_{\frac{j}{2}}\gamma'_{\frac{j+1}{2}}}.$$

3.2. **The joint law of $(|Z_N|, \Im\mathfrak{m} \log Z_N)$.** In this Section, we use the Mellin Fourier transform (1.1) obtained in Section 2 to deduce the joint law of $(|Z_N|, \Im\mathfrak{m} \log Z_N)$. For simplicity, let us write

$$\Delta_N \equiv |Z_N|, \text{and} \;\; I_N \equiv \Im\mathfrak{m} \log Z_N$$

so with this notation formula (1.1) states

$$\mathbb{E}\left[\Delta_N^t e^{isI_N}\right] = \prod_{k=1}^{N} \frac{\Gamma(k)\Gamma(k+t)}{\Gamma\left(k+\frac{t+s}{2}\right)\Gamma\left(k+\frac{t-s}{2}\right)}. \tag{3.2}$$

Lemma 3.1. *Let W_j have density*

$$K_j \cos(v)^{2(j-1)} \mathbb{1}_{(-\pi/2,\pi/2)},$$

where

$$K_j = \frac{2^{2(j-1)}\left((j-1)!\right)^2}{\pi\,(2j-2)!},$$

and let

$$X := \beta_{j,j-1} 2\cos W_j e^{\mathrm{i}W_j}.$$

where all the random variables in sight are independent. Then

$$\mathbb{E}\left[|X|^t e^{\mathrm{i}s\arg X}\right] = \frac{\Gamma(j)\Gamma(j+t)}{\Gamma(j+(t+s)/2)\Gamma(j+(t-s)/2)} \tag{3.3}$$

Proof. By the definition of X, we have that

$$\mathbb{E}\left[|X|^t e^{\mathrm{i}s\arg X}\right] = \\ \mathbb{E}\left[(\beta_{j,j-1})^t\right] K_j \int_{-\pi/2}^{\pi/2} e^{\mathrm{i}sx}\left(e^{\mathrm{i}x}+e^{-\mathrm{i}x}\right)^t \left(e^{\mathrm{i}x}+e^{-\mathrm{i}x}\right)^{2(j-1)} \,\mathrm{d}x$$

By (3.1) we have

$$\mathbb{E}\left[(\beta_{j,j-1})^t\right] = \frac{\Gamma(j+t)\Gamma(2j-1)}{\Gamma(j)\Gamma(2j-1+t)}$$

Note that

$$e^{\mathrm{i}sx}\left(e^{\mathrm{i}x}+e^{-\mathrm{i}x}\right)^{2(j-1)}\left(e^{\mathrm{i}x}+e^{-\mathrm{i}x}\right)^t \\ = \left(1+e^{2\mathrm{i}x}\right)^{j-1+(t+s)/2}\left(1+e^{-2\mathrm{i}x}\right)^{j-1+(t-s)/2}$$

Both terms on the RHS can be expanded as a series in $e^{2\mathrm{i}x}$ or $e^{-2\mathrm{i}x}$ for all x other than $x = 0$. Integrating over x between $-\pi/2$ and $\pi/2$, only the diagonal terms survive, and so

$$\int_{-\pi/2}^{\pi/2} e^{\mathrm{i}sx}\left(e^{\mathrm{i}x}+e^{-\mathrm{i}x}\right)^{2(j-1)}\left(e^{\mathrm{i}x}+e^{-\mathrm{i}x}\right)^{t}\,\mathrm{d}x$$
$$=\sum_{k=0}^{\infty}\frac{\left(-(j-1+(t+s)/2)\right)_k\left(-(j-1+(t-s)/2)\right)_k}{k!k!}$$
$$=H(-(j-1+(t+s)/2),-(j-1+(t-s)/2),1;1)$$

where H is a hypergeometric function. The values of hypergeometric functions at $z=1$ are well known (see, for example [1]), and are given by

$$\frac{\Gamma(2j-1+t)}{\Gamma(j+(t+s)/2)\Gamma(j+(t-s)/2)}$$

and this completes the proof. □

The next Theorem now follows easily from the previous lemma:

Theorem 3.2. *Let $\Delta_N\equiv|Z_N|$ and $I_N\equiv\Im\mathfrak{m}\log Z_N$. Let $(\beta_{j,j-1})_{1\le j\le N}$ be independent beta variables of parameters j and $j-1$ respectively (with the convention that $\beta_{1,0}\equiv 1$). Define $W_1,\dots,W_N$ as independent random variables which are independent of the $(\beta_{j,j-1})_{1\le j\le N}$, with W_j having the density:*

$$\sigma_{2(j-1)}(\mathrm{d}v)=\frac{2^{2(j-1)}\left((j-1)!\right)^2}{\pi\,(2j-2)!}\cos^{2(j-1)}(v)\,\mathbb{1}_{\left(\frac{-\pi}{2},\frac{\pi}{2}\right)}\mathrm{d}v. \tag{3.4}$$

Then, the joint distribution of (I_N,Δ_N) is given by:

$$(I_N,\Delta_N)\overset{\text{law}}{=}\left(\sum_{j=1}^{N}W_j,\prod_{j=1}^{N}\beta_{j,j-1}2\cos W_j\right). \tag{3.5}$$

We now recover a formula obtained in [11] in the study of the relations between the Barnes function and generalized gamma variables. To this end, we need the following elementary lemma whose proof is left to the reader:

Lemma 3.3. *Let V_t be distributed as*

$$\mathbb{P}(V_t\in\mathrm{d}v)=\frac{(2\cos(v))^t}{\pi K_t},\quad |v|<\frac{\pi}{2},$$

with $K_t=\frac{\Gamma(1+t)}{\left(\Gamma\left(1+\frac{t}{2}\right)\right)^2}$. Then

$$\cos(V_t)\overset{\text{law}}{=}\sqrt{\beta_{\frac{t+1}{2},\frac{1}{2}}}.$$

If $W_j\overset{\text{law}}{=}V_{2(j-1)}$ then

$$\cos(W_j)\overset{\text{law}}{=}\sqrt{\beta_{j-\frac{1}{2},\frac{1}{2}}}. \tag{3.6}$$

Proposition 3.4 ([11]). *Let $(\gamma_j)_{1\leq j\leq N}$ and $\left(\gamma_j'\right)_{1\leq j\leq N}$ be sequences of independent gamma(j) variables. Then we have*

$$\prod_{j=1}^{N}\gamma_j \overset{\text{law}}{=} \Delta_N \prod_{j=1}^{N}\sqrt{\gamma_j\gamma_j'}. \tag{3.7}$$

Proof. Considering only the second component in (3.5), and multiplying both sides by $\prod_{j=1}^{N}\gamma_{2j-1}$, and thanks to the beta-gamma algebra, we obtain:

$$\Delta_N\left(\prod_{j=1}^{N}\gamma_{2j-1}\right) \overset{\text{law}}{=} \left(\prod_{j=1}^{N}\gamma_j\right)2^N\left(\prod_{j=1}^{N}\cos\left(W_j\right)\right). \tag{3.8}$$

Now we apply the lemma to the right hand side of the above equality in law to obtain that

$$\left(\prod_{j=1}^{N}\gamma_j\right)2^N\left(\prod_{j=1}^{N}\cos\left(W_j\right)\right) \overset{\text{law}}{=} \left(\prod_{j=1}^{N}\gamma_j\right)2^N\prod_{j=1}^{N}\sqrt{\beta_{j-\frac{1}{2},\frac{1}{2}}}. \tag{3.9}$$

On the other hand, from the duplication formula for the gamma function, we have for any $a>0$

$$\gamma_a \overset{\text{law}}{=} 2\sqrt{\gamma_{\frac{a}{2}}\gamma'_{\frac{a+1}{2}}},$$

thus, on the left hand side of (3.8) we get

$$\Delta_N 2^N\prod_{j=1}^{N}\sqrt{\gamma_{\frac{2j-1}{2}}\gamma_j'} \overset{\text{law}}{=} \Delta_N 2^N\prod_{j=1}^{N}\sqrt{\beta_{j-\frac{1}{2},\frac{1}{2}}\gamma_j\gamma_j'}. \tag{3.10}$$

Now comparing (3.9) and (3.10) we obtain

$$\prod_{j=1}^{N}\gamma_j \overset{\text{law}}{=} \Delta_N \prod_{j=1}^{N}\sqrt{\gamma_j\gamma_j'}.$$

□

Infinitely divisible laws form a very remarkable and well studied family of laws in Probability Theory. It is easily see from Proposition 3.4 that the law of $\log|Z_N|$ is infinitely divisible.

Proposition 3.5. *The law of* $\log|Z_N|$ *is infinitely divisible.*

Proof. It follows from Proposition 3.4 and the fact that logarithm of a gamma variable is infinitely divisible (see, for example, [2]). □

Remark. The law of $\mathfrak{Im}\log Z_N$ is not infinitely divisible since it is a bounded random variable.

4. Central limit theorems

In this section, we give two alternative proofs of the following central limit theorem by Keating and Snaith [7]. The first one from the decomposition in Section 2, the second from the last decomposition in Section 3. The original proof by Keating and Snaith relies on an expansion of formula (1.1) with cumulants.

Theorem 4.1. *Let $Z_N := \det(I_N - V_N)$, where V_N is distributed with the Haar measure on the unitary group $U(N)$. Then,*

$$\frac{\log Z_N}{\sqrt{\frac{1}{2}\log N}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2, \tag{4.1}$$

as $N \to \infty$, with $\mathcal{N}_1$ and $\mathcal{N}_2$ independent standard normal variables.

4.1. **Proof from the decomposition in Section 2.** From Proposition 2.2, we know that

$$\det(I_N - V_N) \overset{\text{law}}{=} \prod_{k=1}^{N} \left(1 + e^{\mathrm{i}\theta_k}\sqrt{\beta_{1,k-1}}\right),$$

with $\theta_1, \dots, \theta_N, \beta_{1,0}, \dots, \beta_{1,N-1}$ independent random variables, the θ_k's uniformly distributed on $[0, 2\pi]$ and the $\beta_{1,j}$'s $(0 \le j \le N-1)$ being beta distributed with parameters 1 and j.

In the following we note

$$X_N := \sum_{i=1}^{N} \log\left(1 + e^{\mathrm{i}\theta_k}\sqrt{\beta_{1,k-1}}\right),$$

with $\log(1+\epsilon)$ defined here as $\sum_{j\ge 1}(-1)^{j+1}\epsilon^j/j$ (this is convergent a.s. because $|e^{\mathrm{i}\theta_k}\sqrt{\beta_{1,k-1}}| < 1$ a.s.).

What we need in order to prove Theorem 4.1 is the following :

(1) first to show that X_N is equal in law to $\log Z_N$, as it is defined in the introduction. It is not so obvious, because the imaginary parts could have a $2k\pi$ difference.
(2) then show that X_N converges to what is expected in (4.1).

Proof for (1). Equation (2.3), stated for a fixed x, is also obviously also true for a trajectory, for example for $x \in [0,1]$,

$$\begin{aligned}&(\det(I_N - xV_N), 0 \le x \le 1)\\ &\qquad \overset{\text{law}}{=} ((1 - f(x, V_{N-1}, M_1))\det(I_{N-1} - xV_{N-1}), 0 \le x \le 1),\end{aligned}$$

with the suitable f from (2.3). Let the logarithm be defined as in the Introduction (ie by continuity from $x = 0$). The previous equation then implies, as f is continuous in x,

$$\log\det(I_N - xV_N) \overset{\text{law}}{=} \log(1 - f(x, V_{N-1}, M_1)) + \log\det(I_{N-1} - xV_{N-1}).$$

One can easily check that $|f(x,V_{N-1},M_1)|<1$ for all $x\in[0,1]$ a.s., so

$$\log(1-f(x,V_{N-1},M_1))=\sum_{j\geq 0}(-1)^{j+1}\frac{f(x,V_{N-1},M_1)^j}{j}$$

for all $x\in[0,1]$ almost surely. In particular, for $x=1$, we get

$$\log\det(I_N-V_N)\overset{\text{law}}{=}\left(\sum_{j\geq 0}(-1)^{j+1}\frac{M_{11}^j}{j}\right)+\log\det(I_{N-1}-V_{N-1}),$$

which gives the expected result by an immediate induction. We have therefore shown that $\log Z_N\overset{\text{law}}{=}X_N$.

Proof for (2). The idea is basically that $\beta_{1,k-1}$ tends in law to a Dirac distribution on 0 as k tends to ∞. So $\log(1+e^{\mathrm{i}\theta_k}\sqrt{\beta_{1,k-1}})$ is well approximated by $e^{\mathrm{i}\theta_k}\sqrt{\beta_{1,k-1}}$, and as this has a distribution invariant by rotation, the central limit theorem will be easily proven from classical results in dimension 1.

More precisely, X_N can be decomposed as

$$X_N=\underbrace{\sum_{k=1}^N e^{\mathrm{i}\theta_k}\sqrt{\beta_{1,k-1}}}_{X_1(N)}-\underbrace{\frac{1}{2}\sum_{k=1}^N e^{2\mathrm{i}\theta_k}\beta_{1,k-1}}_{X_2(N)}+\underbrace{\sum_{j\geq 3}\sum_{k=1}^N\frac{(-1)^{j+1}}{j}\left(e^{\mathrm{i}\theta_k}\sqrt{\beta_{1,k-1}}\right)^j}_{X_3(N)}$$

where all the terms are absolutely convergent. We study these three terms separately.

Clearly $X_1(N)$ has a distribution which is invariant by rotation, so to prove that $\frac{X_1(N)}{\sqrt{\frac12\log N}}\overset{\text{law}}{\longrightarrow}\mathcal{N}_1+\mathrm{i}\mathcal{N}_2$, we only need to prove the following result for the real part :

$$\frac{\sum_{k=1}^N\cos\theta_k\sqrt{\beta_{1,k-1}}}{\sqrt{\frac12\log N}}\overset{\text{law}}{\longrightarrow}\mathcal{N},$$

where $\mathcal{N}$ is a standard normal variable. As $\mathbb{E}(\cos^2(\theta_k)\beta_{1,k-1})=\frac{1}{2k}$, this is a direct consequence of the central limit theorem (our random variables check the Lyapunov condition).

To deal with $X_2(N)$, as $\sum_{k\geq 0}1/k^2<\infty$, there exists a constant $c>0$ such as $\mathbb{E}(|X_2(N)|^2)<c$ for all $N\in\mathbb{N}$. Thus $(X_2(N),N\geq 1)$ is a L^2-bounded martingale, so it converges almost surely. Hence

$$X_2(N)/\sqrt{\tfrac12\log N}\to 0\quad a.s.$$

Finally, for $X_3(N)$, let $Y:=\sum_{j=3}^\infty\sum_{k=1}^\infty\frac1j(\beta_{1,k-1})^{j/2}$. One can easily check that $\mathbb{E}(Y)<\infty$, so $Y<\infty$ a.s., so as $N\to\infty$

$$|X_3(N)|/\sqrt{\tfrac12\log N}<Y/\sqrt{\tfrac12\log N}\to 0\quad a.s.$$

Gathering all these convergences, we get the expected result :

$$\frac{X_N}{\sqrt{\frac{1}{2}\log N}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2,$$

with the notations of Theorem 4.1.

4.2. **Proof from the decomposition in section 3.** We shall give here a very simple proof of the central limit theorem for $\frac{\log Z_N}{\sqrt{\frac{1}{2}\log N}}$ based on the decomposition into sums of independent random variables and a classical version of the multidimensional central limit theorem.

From Theorem 3.2, we have:

$$(\mathfrak{Im}\log Z_N, \log|Z_N|) \stackrel{\text{law}}{=} \left(\sum_{j=1}^{N} W_j, \sum_{j=1}^{N} T_j\right). \tag{4.2}$$

where

$$T_j = \log(\beta_{j,j-1} 2\cos W_j). \tag{4.3}$$

Now, from the discussion preceding Theorem 3.2, we have for $s > -1$ and $t > -1$:

$$\mathbb{E}\left[e^{\mathrm{i}sW_j}\right] = \frac{\Gamma(j)^2}{\Gamma\left(j+\frac{s}{2}\right)\Gamma\left(j-\frac{s}{2}\right)}, \tag{4.4}$$

$$\mathbb{E}\left[e^{tT_j}\right] = \frac{\Gamma(j)\Gamma(j+t)}{\Gamma\left(j+\frac{t}{2}\right)^2}. \tag{4.5}$$

From these Fourier transforms, one can easily deduce the moments or the cumulants of all orders for W_j and T_j (see [15, **?**, **?**] for definition of cumulants and their relations with moments) by taking successive derivatives at 0. For our purpose, we will only need the first three moments or cumulants. Since the calculation of the derivatives have already been done in [7], we will only recap them here. Let us call $Q_{j,k}$ the k-th cumulant of T_j and $R_{j,k}$ the k^{th} cumulant of W_j. Then we have

$$Q_{j,k} = \frac{2^{k-1}-1}{2^{k-1}}\psi^{(k-1)}(j)$$

and

$$R_{j,k} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{(-1)^{k/2+1}}{2^{k-1}}\psi^{(k-1)}(j) & \text{if } k \text{ is even} \end{cases}$$

where

$$\psi^{(k)}(z) = \frac{\mathrm{d}^{k+1}\log\Gamma(z)}{dz^{k+1}}$$

are the polygamma functions. Now, since the cumulants of a sum of independent random variables are the sum of the cumulants, we can easily obtain that the cumulants of $\sum_{j=1}^{N} T_j$ and $\sum_{j=1}^{N} W_j$ are respectively

$$\frac{2^{k-1}-1}{2^{k-1}} \sum_{j=1}^{N} \psi^{(k-1)}(j)$$

and

$$\begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{(-1)^{k/2+1}}{2^{k-1}} \sum_{j=1}^{N} \psi^{(k-1)}(j) & \text{if } k \text{ is even} \end{cases}.$$

Moreover, we have the following expansion of the polygamma function (see, for example Corollary 1.4.5 of [1]):

$$\psi(z) \sim \log z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}} \tag{4.6}$$

and

$$\psi^{(k)}(z) = (-1)^{k-1} \left[\frac{(k-1)!}{z^k} + \frac{k!}{2z^{k+1}} + \sum_{n=0}^{\infty} B_{2n} \frac{(2n+k-1)!}{(2n)! z^{2n+k}} \right] \tag{4.7}$$

for $|z| \to \infty$ and $|\arg z| < \pi$, and where the B_{2n} are the Bernoulli numbers. We deduce from (4.6) that the variances of $\mathfrak{Re} \log Z_N$ and $\mathfrak{Im} \log Z_N$ (which are centered) are finite and both asymptotic to $\frac{1}{2} \log N$ as $N \to \infty$.

Now we state the central limit theorem we shall apply (we follow page 87 of Strook [19]). We assume that (X_n) is a sequence of independent, square integrable $\mathbb{R}^{\ell}$ valued random variables defined on the same probability space. Further we will assume that X_n has mean 0 and strictly positive covariance $\operatorname{cov}(X_n)$. Finally we set:

$$S_n = \sum_{m=1}^{n} X_m, \quad C_n := \operatorname{cov}(S_n) = \sum_{m=1}^{n} \operatorname{cov}(X_m)$$

and

$$\Sigma_n = (\det(C_n))^{\frac{1}{2\ell}} \quad \text{and } \widehat{S}_n = \frac{S_n}{\Sigma_n}.$$

Theorem 4.2 (Multidimensional Central Limit Theorem, [19] p. 88)**.** *Assume that*

$$A := \lim_{n\to\infty} \frac{C_n}{\Sigma_n^2}$$

exists and that

$$\lim_{n\to\infty} \frac{1}{\Sigma_n^2} \sum_{m=1}^{n} \mathbb{E}\left[|X_m|^2 \mathbb{1}_{|X_m| \geq \varepsilon \Sigma_n} \right] = 0 \tag{4.8}$$

for every $\varepsilon > 0$. Then the vector $\widehat{S}_n$ converges in law to a Gaussian vector with mean 0 and covariance matrix A.

Now we can prove Theorem 4.1 :

$$\frac{Z_N}{\sqrt{\frac{1}{2}\log N}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2.$$

Indeed, let us consider the Lyapounov exponents associated with (T_n) and (W_n):

$$L_N = \frac{1}{s_N^{3/2}} \sum_{n=1}^{N} \mathbb{E}\left[|T_n|^3\right],$$

and

$$L'_N = \frac{1}{\sigma_N^{3/2}} \sum_{n=1}^{N} \mathbb{E}\left[|W_n|^3\right],$$

where $s_N^2 = \sum_{j=1}^{N} \mathbb{E}\left[T_j^2\right]$ and $\sigma_N^2 = \sum_{j=1}^{N} \mathbb{E}\left[W_j^2\right]$. From the expressions of the cumulants, we have:

$$s_N^2 = \sigma_N^2 = \frac{1}{2}\sum_{j=1}^{N} \psi'(j) \sim \frac{1}{2}\log N.$$

It is not hard to see, using the expression for the cumulants or the density of the beta variables and the W_j, that the series $\sum_{n=1}^{\infty} \mathbb{E}\left[|T_n|^3\right]$ and $\sum_{n=1}^{\infty} \mathbb{E}\left[|W_n|^3\right]$ both converge. Hence $L_N \to 0$ and $L'_N \to 0$ as $N \to \infty$, and consequently (4.8) holds and the result follows from an application of the Multidimensional Central Limit Theorem of Theorem 4.2.

5. Iterated logarithm law

In this Section, we give some iterated logarithm law for both the real and imaginary parts of the characteristic polynomial. Again, this can be done due to the decomposition given in Theorem 3.2.

We first need some information about the rate of convergence in the central limit theorem.

5.1. Further results about the rate of convergence.

With the representation in Theorem 3.2 it is possible to obtain uniform and non-uniform estimates on the rate of convergence in the central limit theorem, using the Berry-Essen's inequalities (see [15] or [**?**]):

Theorem 5.1. *Let $X_1, \dots, X_n$ be independent random variables such that $\mathbb{E}[X_j] = 0$, and $\mathbb{E}\left[|X_j|^3\right] < \infty$. Put $\sigma_j^2 = \mathbb{E}\left[X_j^2\right]$; $B_n = \sum_{j=1}^{N} \sigma_j^2$; $F_n(x) = \mathbb{P}\left[B_n^{-1/2} \sum_{j=1}^{n} X_j \le x\right]$ and*

$$L_n = \frac{1}{B_n^{3/2}} \sum_{j=1}^{n} \mathbb{E}\left[|X_j|^3\right].$$

Then there exist two constants A and C not depending on n such that the following uniform and non uniform estimates hold:

$$\sup_x |F_n(x) - \Phi(x)| \leq AL_n \tag{5.1}$$

and

$$|F_n(x) - \Phi(x)| \leq \frac{CL_n}{(1+|x|)^3}. \tag{5.2}$$

Now, applying the above theorem to the variables (T_j) and (W_j) we obtain:

Proposition 5.2. *The following estimates on the rate of convergence in the central limit theorem for the real and imaginary parts of the characteristic polynomial hold:*

$$\left|\mathbb{P}\left[\frac{\mathfrak{Re}\log Z_N}{\sqrt{\frac{1}{2}\log N}} \leq x\right] - \Phi(x)\right| \leq \frac{C}{(\log N)^{3/2}(1+|x|)^3} \tag{5.3}$$

$$\left|\mathbb{P}\left[\frac{\mathfrak{Im}\log Z_N}{\sqrt{\frac{1}{2}\log N}} \leq x\right] - \Phi(x)\right| \leq \frac{C}{(\log N)^{3/2}(1+|x|)^3} \tag{5.4}$$

where C is a constant.

5.2. **An iterated logarithm law.** We first state a theorem of Petrov, [13, 14].

Theorem 5.3 (Petrov). *Let $X_1, X_2, \dots$ be independent random variables such that $\mathbb{E}[X_j] = 0$, and $\sigma_j^2 = \mathbb{E}\left[X_j^2\right] < \infty$. Set $B_n = \sum_{j=1}^N \sigma_j^2$; $F_n(x) = \mathbb{P}\left[B_n^{-1/2}\sum_{j=1}^n X_j \leq x\right]$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\mathrm{d}t$. If the conditions*

(1) $B_n \to \infty$;
(2) $\frac{B_{n+1}}{B_n} \to 1$;
(3) $\sup_x |F_n(x) - \Phi(x)| = \mathcal{O}\left((\log B_n)^{-1-\delta}\right)$,

are satisfied for some $\delta > 0$, then

$$\limsup \frac{S_n}{\sqrt{2B_n \log\log B_n}} = 1 \quad a.s. \tag{5.5}$$

Remark. If the conditions of the theorem are satisfied, then we also have:

$$\liminf \frac{S_n}{\sqrt{2B_n \log\log B_n}} = -1 \quad \text{a.s.}$$

Before using Theorem 5.3 for the real and imaginary parts of $\log Z_N$, we need to give the explicit meaning of the "almost sure convergence" for matrices with different sizes.

Imagine that in Proposition 2.1 we choose for M the symmetry which transforms e_1 (the first vector of the basis) into M_1, a random vector of

$\mathscr{S}_{\mathbb{C}}^{n+1}$. Consider the set $O = \mathscr{S}_{\mathbb{C}}^1 \times \mathscr{S}_{\mathbb{C}}^2 \times \mathscr{S}_{\mathbb{C}}^3 \dots$ endowed with the measure $\nu_1 \times \nu_2 \times \nu_3 \dots$, where ν_k is the uniform measure on the sphere $\mathscr{S}_{\mathbb{C}}^k$ (this can be a probability measure by defining the measure of a set as the limiting measure of the finite-dimensional cylinders). Consider the application f which transforms $\omega \in O$ into an element of $U(1) \times U(2) \times U(3) \dots$ with successive iterations of the Proposition 2.1. Then $\Omega = \mathfrak{Im}(f)$ is naturally endowed with a probability measure $\mu_U = \mathfrak{Im}(\nu)$, and the marginal distribution of μ_U on the k^{th} coordinate is the Haar measures on $U(k)$.

Let g be a function of a unitary matrix U, no matter the size of U (e.g. $g = \det(Id - U)$). The introduction of the set Ω with measure μ_U allows us to define the "almost sure" convergence of $(g(U_k), k \geq 0)$, where $(U_k)_{k\geq 0} \in \Omega$. This is, for instance, the sense of the "a.s" in the following iterated logarithm law.

Proposition 5.4. *The following almost sure convergence (defined previously) holds :*

$$\limsup \frac{\mathfrak{Re} \log Z_N}{\sqrt{\log N \log\log\log N}} = 1, \tag{5.6}$$

$$\limsup \frac{\mathfrak{Im} \log Z_N}{\sqrt{\log N \log\log\log N}} = 1. \tag{5.7}$$

Remark. The representations in law as sums of independent random variables we have obtained could as well be used to obtain all sorts of refined large and moderate deviations estimates for the characteristic polynomial.

6. Same results in the orthogonal case.

The Mellin Fourier transform for $Z := \det(I_N - M)$ found in [7] and [8] by Keating and Snaith, using the Selberg integrals, are, for the unitary group $U(N)$

$$\mathbb{E}\left[|Z|^t e^{\mathrm{i}s \arg Z}\right] = \prod_{k=1}^{N} \frac{\Gamma(k)\,\Gamma(k+t)}{\Gamma\left(k + \frac{t+s}{2}\right)\Gamma\left(k + \frac{t-s}{2}\right)}; \tag{6.1}$$

and for the special orthogonal group $SO(2N)$

$$\mathbb{E}\left[Z^t\right] = 2^{2Nt} \prod_{k=1}^{N} \frac{\Gamma(N+k-1)\,\Gamma\left(t+k-\frac{1}{2}\right)}{\Gamma\left(k-\frac{1}{2}\right)\Gamma(t+k+N-1)}. \tag{6.2}$$

Formula (6.1) was directly proven in section 2. Here we show that such a probabilistic proof still holds for formula (6.2).

Let

$$\mathscr{S}_{\mathbb{R}}^N := \{(r_1, \dots, r_N) \in \mathbb{R}^N : |r_1|^2 + \dots + |r_N|^2 = 1\}$$

and $\mu_{O(N)}$ be the Haar measure on $O(N)$. Then in analogy to Proposition 2.1, we have:

Proposition 6.1. *Let $M \in O(N+1)$ be chosen such that its first column M_1 is uniformly distributed on $\mathscr{S}_{\mathbb{R}}^{N+1}$. Let $O_N \in O(N)$ be chosen independently of M according to the Haar measure $\mu_{O(N)}$. Then the matrix*

$$O_{N+1} := M \begin{pmatrix} 1 & 0 \\ 0 & O_N \end{pmatrix}$$

is distributed with the Haar measure $\mu_{O(N+1)}$.

The proof is essentially the same as the proof for Proposition 2.1. If we choose for M a symmetry transforming e_1 in a uniformly chosen vector of $\mathscr{S}_{\mathbb{R}}^{(n+1)}$, we transform a random element of $SO(n)$ into an element of $O(n+1)$ with determinant -1, and reciprocally. As a consequence, the following result, analogous of Proposition 2.2, can easily be shown.

Corollary 6.2. *Let $SO \in SO(2n)$ be distributed with the Haar measure $\mu_{SO(2n)}$. Then*

$$\det(I_{2n} - SO) \overset{\text{law}}{=} 2 \prod_{k=2}^{2n} \left(1 + \epsilon_k \sqrt{\beta_{\frac{1}{2}, \frac{k-1}{2}}} \right),$$

with $\epsilon_1, \dots, \epsilon_{2n}, \beta_{1/2,1/2}, \dots, \beta_{1/2,(2n-1)/2}$ independent random variables such that $\mathbb{P}(\epsilon_k = 1) = \mathbb{P}(\epsilon_k = -1) = 1/2$, and the β's being beta distributed with the indicated parameters.

Remark. A direct calculation with the suitable change of variables shows that

$$1 + \epsilon_k \sqrt{\beta_{\frac{1}{2}, \frac{k-1}{2}}} \overset{\text{law}}{=} 2\beta_{\frac{k-1}{2}, \frac{k-1}{2}},$$

from which formula (6.2) can be easily recovered.

Remark. The same reasoning can be applied to many other groups such as $\mathbb{H}(n)$, the set of $n \times n$ matrices H_n on the field of quaternions with $\overline{H_n}^T H_n = I_n$. The symplectic group requires some additional work, to appear in a future paper.

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