

Bulk Universality of General β -Ensembles with Non-convex Potential

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Abstract

We prove the bulk universality of the β -ensembles with non-convex regular analytic potentials for any $\beta > 0$. This removes the convexity assumption appeared in the earlier work [6]. The convexity condition enabled us to use the logarithmic Sobolev inequality to estimate events with small probability. The new idea is to introduce a “convexified measure” so that the local statistics are preserved under this convexification.

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1 Introduction and the main results

The classical invariant ensembles of random matrices are given by probability measures of the form $e^{-N\beta\text{Tr}V(H)/2}$ where N is the size of the matrix H and V is a real valued potential. The parameter $\beta = 1, 2, 4$ is determined by the symmetry type of the matrix, corresponding respectively to the classical orthogonal, unitary or symplectic ensemble. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N) \in \Sigma_N$ be the eigenvalues of H in increasing order, where $\Sigma_N \subset \mathbb{R}^N$ denotes the simplex determined by $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$. It is well-known that the probability distribution of the ordered eigenvalues on Σ_N is given by

$$\mu_{\beta,V}^{(N)} = \mu^{(N)}(d\lambda) = \frac{1}{Z_N} e^{-\beta N \mathcal{H}(\lambda)} d\lambda, \quad \mathcal{H}(\lambda) = \sum_{k=1}^N \frac{1}{2} V(\lambda_k) - \frac{1}{N} \sum_{1 \leq i < j \leq N} \log(\lambda_j - \lambda_i). \quad (1.1)$$

For non-classical values of $\beta > 0$, i.e., $\beta \notin \{1, 2, 4\}$, one can still consider the measure (1.1) on Σ_N , but in general there is no simple natural matrix model producing this measure except for the Gaussian case, $V(x) = x^2$, which corresponds to a tri-diagonal random matrix [13, 26]. We will view $\mu = \mu^{(N)}$ as a Gibbs measure of particles in \mathbb{R} with a logarithmic interaction, where the parameter $\beta > 0$ is interpreted as the inverse temperature. We will refer to the variables λ_j as particles or points and the system is called log-gas or general β -ensemble.

The universality conjecture asserts that the eigenvalue gap distributions in the bulk depend only on β and are independent of the potential V . For classical ensembles, the eigenvalue correlation functions can be explicitly expressed in terms of polynomials orthogonal to the measure $e^{-\beta V(x)/2}$. Thus the analysis of the correlation functions relies heavily on the asymptotic properties of the corresponding orthogonal polynomials. This approach, initiated by Dyson, Gaudin and Mehta (see [21] for a review) was the starting point for all results on classical universality. Precise analysis on orthogonal polynomials for general class of weight functions was made possible by the Riemann-Hilbert approach [5, 11, 12]. There are also methods independent of the Riemann-Hilbert approach, see, e.g., [20, 22, 23]. The universality for $\beta = 2$ was proved for very general potential. For $\beta = 1, 4$ [10, 18, 25] it was proved for analytic V with some additional conditions. A summary of recent developments can be found in [2, 9, 10, 24].

For non-classical values of β , i.e., $\beta \notin \{1, 2, 4\}$, there is no simple expression of the correlation functions in terms of orthogonal polynomials. In [6], we initiated a new approach to prove bulk universality for all $\beta > 0$ and strictly convex V . The method was based on estimating correlation functions by local Dirichlet form and the main ingredients consist of the following two steps:

Step 1. Rigidity of eigenvalues. This establishes that the location of the eigenvalues are not too far from their classical locations determined by the equilibrium density $\rho(s)$.

Step 2. Uniqueness of local Gibbs measures with logarithmic interactions. With the precision of eigenvalue location estimates from Step 1 as an input, the eigenvalue gap distributions are shown to be given by the corresponding Gaussian ones. (We will take the uniqueness of the gap distributions as our definition of the uniqueness of Gibbs state.)

Our goal is to extend this result to the non-convex case. It was emphasized in [6] that the convexity of the potential V was used only in Step 1. So in order to apply this method, it suffices to prove the rigidity estimate which we now introduce.

We will assume that the potential V is real analytic function in \mathbb{R} such that its second derivative is bounded below, i.e. we have

$$\inf_{x \in \mathbb{R}} V''(x) \geq -2W \quad (1.2)$$

for some constant $W \geq 0$, and

$$V(x) > (2 + \alpha) \ln(1 + |x|), \quad (1.3)$$

for some $\alpha > 0$, if $|x|$ is large enough. It is known [7] that under these (in fact, even weaker) conditions the measure is normalizable, $Z_N < \infty$. Moreover, the averaged density of the empirical spectral measure, defined as

$$\rho_1^{(N)}(\lambda) = \rho_1^{(N,\beta,V)}(\lambda) := \mathbb{E}_{\mu^{(N)}} \frac{1}{N} \sum_{j=1}^N \delta(\lambda - \lambda_j)$$

converges weakly to a continuous function ρ , the equilibrium density, with compact support. We additionally assume that $\rho(s)$ is supported on a single interval $[A, B]$, and that V is *regular* in the sense of [19]. We recall that V is regular if its equilibrium density ρ is positive on (A, B) and vanishes like a square root at each of the endpoints of $[A, B]$, that is

$$\begin{aligned} \rho(t) &= s_A \sqrt{t - A} (1 + O(t - A)), \quad t \rightarrow A^+, \\ \rho(t) &= s_B \sqrt{B - t} (1 + O(B - t)), \quad t \rightarrow B^-, \end{aligned} \quad (1.4)$$

for some constants $s_A, s_B > 0$.

Remark 1.1 *This regularity assumption is not a strong constraint: [19] proves that the regular potentials V are a dense and open subset of the potentials for the topology induced by the distance*

$$d(V, W) = \sum_{j=0}^3 \sum_{k=1}^{\infty} 2^{-k} \frac{\|V^{(j)} - W^{(j)}\|_{L^\infty[-k,k]}}{1 + \|V^{(j)} - W^{(j)}\|_{L^\infty[-k,k]}} + \sum_{k=1}^{\infty} 2^{-k} \frac{|G_k(V) - G_k(W)|}{1 + |G_k(V) - G_k(W)|},$$

where $G_k(V) = \inf_{|x|>k} V(x) / \log |x|$.

In this paper, we are interested in the usual n -point correlation functions, generalizing $\rho_1^{(N)}$, and defined by

$$\rho_n^{(N)}(x_1, \dots, x_n) = \int_{\mathbb{R}^{N-n}} \tilde{\mu}(x) dx_{n+1} \dots dx_N, \quad (1.5)$$

where $\tilde{\mu}$ is the symmetrized version of μ given in (1.1) but defined on \mathbb{R}^N instead of the simplex Σ_N :

$$\tilde{\mu}^{(N)}(d\lambda) = \frac{1}{N!} \mu(d\lambda^{(\sigma)}),$$

where $\lambda^{(\sigma)} = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(N)})$, with $\lambda_{\sigma(1)} < \dots < \lambda_{\sigma(N)}$.

In the following, we omit the superscript N and we will write μ for $\mu^{(N)}$. We will use \mathbb{P}_μ and \mathbb{E}_μ to denote the probability and the expectation with respect to μ . Let the *classical position* γ_k be defined by

$$\int_{-\infty}^{\gamma_k} \rho(s) ds = \frac{k}{N}. \quad (1.6)$$

Finally, we introduce the notation $\llbracket p, q \rrbracket = [p, q] \cap \mathbb{Z}$ for any real numbers $p < q$.

It is known that the particles are rigid, i.e. they cannot be far from their classical locations: for any $\varepsilon > 0$ there are positive constants c_1, c_2 such that, for all $N \geq 1$,

$$\mathbb{P}_\mu (\exists k \in \llbracket 1, N \rrbracket \mid |\lambda_k - \gamma_k| \geq \varepsilon) \leq c_1 e^{-c_2 N}. \quad (1.7)$$

For eigenvalues in the bulk, (1.7) follows from the large deviations for the empirical spectral measure with speed N^2 [2, 4], and for the extreme eigenvalues the large deviations principle with speed N was proved in [2], Theorem 2.6.6, up to a condition on the partition function that follows from Theorem 1 (iii) in [25].

The main technical result of this paper is to prove that rigidity holds for the measure μ at the optimal scale $1/N$ in the bulk in the following sense. This theorem extends our rigidity result in [6] to non-convex potential V .

Theorem 1.1 (Rigidity estimate in the bulk) *Let V be real analytic, regular with equilibrium density supported on a single interval $[A, B]$, and satisfy (1.2), (1.3). Take any $\alpha > 0$ and $\varepsilon > 0$. Then there are constants $\delta, c_1, c_2 > 0$ such that for any $N \geq 1$ and $k \in \llbracket \alpha N, (1 - \alpha)N \rrbracket$,*

$$\mathbb{P}_\mu (|\lambda_k - \gamma_k| > N^{-1+\varepsilon}) \leq c_1 e^{-c_2 N^\delta}. \quad (1.8)$$

Our main result on the universality is the following theorem:

Theorem 1.2 (Bulk universality) *Let V be real analytic, regular with equilibrium density supported on a single interval $[A, B]$, and satisfy (1.2), (1.3). Then for any $\beta > 0$ the bulk universality holds for the β -ensemble $\mu = \mu_{\beta, V}$. More precisely, for any $E \in (A, B)$ and $|E'| < 2$, for any smooth test functions O with compact support and for any $0 < k \leq \frac{1}{2}$, we have, with $s := N^{-1+k}$, that*

$$\lim_{N \rightarrow \infty} \int d\alpha_1 \cdots d\alpha_n O(\alpha_1, \dots, \alpha_n) \left[\int_{E-s}^{E+s} \frac{dx}{2s} \frac{1}{\rho(E)^n} \rho_n^{(N)} \left(x + \frac{\alpha_1}{N\rho(E)}, \dots, x + \frac{\alpha_n}{N\rho(E)} \right) - \int_{E'-s}^{E'+s} \frac{dx}{2s} \frac{1}{\rho_{sc}(E')^n} \rho_{\text{Gauss}, n}^{(N)} \left(x + \frac{\alpha_1}{N\rho_{sc}(E')}, \dots, x + \frac{\alpha_n}{N\rho_{sc}(E')} \right) \right] = 0.$$

Here $\rho_{sc}(E) = \frac{1}{2\pi} \sqrt{4 - E^2}$ is the Wigner semicircle law and $\rho_{\text{Gauss}, n}^{(N)}$ are the correlation functions of the Gaussian β -ensemble, i.e. with $V(x) = x^2$.

Theorem 1.2 follows immediately from the rigidity estimates, (1.7), (1.8), and the uniqueness of local Gibbs measure, i.e., Theorem 2.1 and Corollary 2.2 in [6]. We note that the proof of the latter results in Section 4 of [6] uses only the rigidity estimate, given in Theorem 3.1 of [6], as an input. Once the rigidity estimate is proven, the rest of the argument is identical and we will not repeat it here.

The rest of this paper is devoted to the proof of Theorem 1.1. After some initial estimates concerning the large deviations regime and global smooth linear statistics (Section 2), the proof consists in the following steps. First we compare μ to some *convexified measures* ν (Section 3); the Hamiltonian \mathcal{H}_ν of ν differs from that of μ mainly by some properly chosen linear statistics of the λ_i 's, allowing \mathcal{H}_ν to be convex. Despite this change in convexity, we will prove that the two measures μ and ν have the same subexponentially small probability events. This step is the main extra ingredient allowing one to generalize the rigidity estimate obtained in [6]. Then by a self-improving method, this measure ν (together with μ) is proved to have rigidity till the optimal scale, thanks to comparisons with locally constrained versions of ν (Section 4).

2 Preliminary results

2.1 Equilibrium measure, large deviations

For analytic potential V satisfying the asymptotic growth condition (1.3), the equilibrium measure $\rho(s)ds$ associated with $(\mu^{(N)})_{N \geq 0}$ can be defined as the unique minimizer (in the set of probability measures on \mathbb{R} endowed with the weak topology) of the functional

$$I(\nu) = \int V(t)d\nu(t) - \iint \log|t-s|d\nu(s)d\nu(t)$$

if $\int V(t)d\nu(t) < \infty$, and $I(\nu) = \infty$ otherwise. Moreover, if one assumes that ρ is supported on a single interval $[A, B]$ and regular in the sense of the previous section, ρ has the following properties:

(a) This equilibrium measure satisfies

$$\frac{1}{2}V'(t) = \int \frac{\rho(s)ds}{t-s}. \quad (2.1)$$

for any $t \in (A, B)$.

(b) For any $t \in [A, B]$,

$$\rho(t) dt = \frac{1}{\pi}r(t)\sqrt{(t-A)(B-t)}\mathbb{1}_{[A,B]} dt, \quad (2.2)$$

where r can be extended into an analytic function in \mathbb{C} satisfying

$$r(z) = \frac{1}{2\pi} \int_A^B \frac{V'(z) - V'(t)}{z-t} \frac{dt}{\sqrt{(t-A)(B-t)}}. \quad (2.3)$$

In order to have the density supported strictly in a compact interval, for given $\kappa > 0$, define the following variant of $\mu^{(N)}$ conditioned to have all particles in $[A - \kappa, B + \kappa]$:

$$\mu^{(N,\kappa)}(d\lambda) = \frac{1}{Z_{N,\kappa}} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \prod_{k=1}^N e^{-N \frac{\beta}{2} V(\lambda_k)} \mathbb{1}_{\lambda_k \in [A-\kappa, B+\kappa]} d\lambda_1 \dots d\lambda_N. \quad (2.4)$$

In this paper we will choose κ to be small. This choice differs from [6] where, instead of $[A - \kappa, B + \kappa]$, we restricted the particles to $[-R, R]$ for a very large R . The smaller interval is needed here because we need r to be positive on the support of $\mu^{(N,\kappa)}$ in the proof of Lemma 2.2. Unlike in the case of convex V where r is known to have no real zero at all, for the non-convex regular case we only know that r is nonzero in the interval $[A, B]$. By continuity, it is also nonzero in $[A - \kappa, B + \kappa]$ for some small κ .

Let $\rho_k^{(N,\kappa)}$ denote the correlation functions of the measure $\mu^{(N,\kappa)}$. Then Lemma 1 in [7] states that under condition (1.3), for some large enough κ there exists some $c > 0$, depending only on V , such that for any $x_1, \dots, x_k \in [A - \kappa, B + \kappa]$, we have

$$\left| \rho_k^{(N,\kappa)}(x_1, \dots, x_k) - \rho_k^{(N)}(x_1, \dots, x_k) \right| \leq \rho_k^{(N,\kappa)}(x_1, \dots, x_k) e^{-cN}, \quad (2.5)$$

and for $x_1, \dots, x_j \notin [A - \kappa, B + \kappa]$, $x_{j+1}, \dots, x_k \in [A - \kappa, B + \kappa]$,

$$\rho_k^{(N)}(x_1, \dots, x_k) \leq e^{-cN \sum_{i=1}^j \log|x_i|}. \quad (2.6)$$

The estimates (2.5) and (2.6) actually also hold for arbitrarily small fixed $\kappa > 0$ thanks to the large deviations estimates (1.7).

2.2 Linear statistics

The following lemma was essentially proven in [25] (for the variance of linear statistics).

Lemma 2.1 *For any function ϕ with $\|\phi\|_\infty + \|\phi'\|_\infty + \|\phi''\|_\infty < \infty$, there is a constant $c > 0$ depending only on V and ϕ (one can choose $c = O(\|\phi\|_\infty + \|\phi'\|_\infty + \|\phi''\|_\infty)$) such that, for any $N \geq 1$ and $s > 0$,*

$$\mathbb{P}_\mu \left(\left| \sum_{i=1}^N \phi(\lambda_i) - N \int_{\mathbb{R}} \rho(u) \phi(u) du \right| > s \right) \leq e^{-cs/\log N}.$$

Proof. Without loss of generality, we can assume that ϕ is compactly supported (thanks to large deviation estimates such as (2.6)). We know from Shcherbina, equation (2.22) in [25], that for the Stieltjes transforms, i.e. $g(u) = 1/(z - u)$, there is a constant $c > 0$ depending only on V and g (one can choose $c = O(\|g^{(4)}\|_\infty)$) such that, for any $N \geq 1$,

$$\left| \mathbb{E}_{\mu_h} \left(\sum_{i=1}^N g(\lambda_i) - N \int_{\mathbb{R}} \rho(u) g(u) du \right) \right| \leq c \log N, \quad (2.7)$$

where μ_h is obtained by replacing V by $V + \frac{h}{N}$ in the definition of μ , and h is for example any N -independent smooth compactly supported function. We will now prove that this implies that (2.7) actually holds when replacing g by any smooth compactly supported ϕ , for example by a Helffer-Sjöstrand type argument, similar to Lemma 2.3. We can now apply formula (B.13) in [15] for the signed measure $\tilde{\rho} = \rho_1^{(N, \mu_h)} - \rho$, with Stieltjes transform S , where $\rho_1^{(N, \mu_h)}$ is the one-point correlation function of μ_h . We obtain

$$\left| \int_{-\infty}^{\infty} \phi(\lambda) \tilde{\rho}(\lambda) d\lambda \right| \leq C \left| \iint y \phi''(x) \chi(y) \operatorname{Im} S(x + iy) dx dy \right| \quad (2.8)$$

$$+ C \iint (|\phi(x)| + |y| |\phi'(x)|) |\chi'(y)| |S(x + iy)| dx dy, \quad (2.9)$$

for some universal $C > 0$, and where χ is a smooth cutoff function with support in $[-1, 1]$, with $\chi(y) = 1$ for $|y| \leq 1/2$ and with bounded derivatives. Note that χ' is supported on $1/2 < |y| < 1$ and ϕ, ϕ' on compact sets, and that S is uniformly $O\left(\frac{\log N}{N}\right)$ on this compact integration domain, by (2.7), so the term (2.9) is easily bounded by $O(\|\phi\|_\infty + \|\phi'\|_\infty) \frac{\log N}{N}$. Concerning the term (2.8), an easy calculation yields the bound $\frac{d}{dy}(y \operatorname{Im} S) = O(1/y)$, so integrating from 1 to y we get $|y \operatorname{Im} S(x + iy)| = O(|\log y|) \frac{\log N}{N}$, which is integrable, so (2.8) is $O(\log N/N)$ as well, finally proving that (2.7) holds when replacing g by ϕ .

Following now Lemma 1 in [25], consider

$$Z_N(t) = \mathbb{E}_\mu \left(\exp \left(\frac{t}{\log N} \left(\sum_{i=1}^N \phi(\lambda_i) - N \int_{\mathbb{R}} \rho(s) \phi(s) ds \right) \right) \right).$$

Then obviously $\frac{d^2}{dt^2} \log Z_N(t) \geq 0$, so

$$\begin{aligned} \log Z_N(t) - \log Z_N(0) &\leq |t| \frac{d}{dt} \log Z_N(t) \\ &= \frac{|t|}{\log N} \mathbb{E}_{\mu_{t\phi/\log N}} \left(\sum_{i=1}^N \phi(\lambda_i) - N \int_{\mathbb{R}} \rho(s) \phi(s) ds \right), \end{aligned}$$

so using (2.7) we get that $Z_N(t) \leq e^{c|t|}$, from which Lemma 2.1 easily follows. \square

2.3 Analysis of the loop equation

This section analyzes the loop equation (2.10) in the following Lemma 2.2. Its proof is very similar to [6] except that, instead of the logarithmic Sobolev inequality which was valid only for convex V , we will use Lemma 2.1. Furthermore, since the support of the restricted measure $\mu^{(N,\kappa)}$ has changed, the integration contours in (2.16) are chosen slightly differently from those in [6].

In the form presented here, we follow closely the proof in [25]. We now introduce some notations needed in the proof.

- m_N is the Stieltjes transform of $\rho_1^{(N)}(s)ds$, evaluated at some z with $\text{Im}(z) > 0$, and m its limit:

$$m_N(z) = \mathbb{E}_\mu \left(\frac{1}{N} \sum_{k=1}^N \frac{1}{z - \lambda_i} \right) = \int_{\mathbb{R}} \frac{1}{z - t} \rho_1^{(N)}(t) dt, \quad m(z) = \int_{\mathbb{R}} \frac{1}{z - t} \rho(t) dt.$$

- $s(z) = -2r(z)\sqrt{(A-z)(B-z)}$, where the square root is defined such that

$$f(z) = \sqrt{(A-z)(B-z)} \sim z \quad \text{as } z \rightarrow \infty;$$

- $b_N(z)$ is defined by

$$b_N(z) = \int_{\mathbb{R}} \frac{V'(z) - V'(t)}{z - t} (\rho_1^{(N)} - \rho)(t) dt;$$

- finally, $c_N(z) = \frac{1}{N^2} k_N(z) + \frac{1}{N} \left(\frac{2}{\beta} - 1 \right) m'_N(z)$, where

$$k_N(z) = \text{var}_\mu \left(\sum_{k=1}^N \frac{1}{z - \lambda_i} \right).$$

Here the var of a complex random variable denotes $\text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$, i.e. without absolute value unlike the usual variance. Note that $|\text{var}(X)| \leq \mathbb{E}(|X - \mathbb{E}(X)|^2)$.

The loop equation (see [16, 17, 25] for various proofs) is

$$(m_N - m)^2 + s(m_N - m) + b_N = c_N. \quad (2.10)$$

In the regime where $|m_N - m|$ is small, we can neglect the quadratic term. The term b_N is the same order as $|m_N - m|$ and is difficult to treat. As observed in [1, 25], for analytic V (hence analytic b_N), this term vanishes when we perform a contour integration. So we have roughly the relation

$$(m_N - m) \sim \frac{1}{N^2} \text{var}_\mu \left(\sum_{k=1}^N \frac{1}{z - \lambda_k} \right), \quad (2.11)$$

where we dropped the less important error involving $m'_N(z)/N$ due to the extra $1/N$ factor. With no convexity assumption on V , the difficulty will be to estimate the above variance to immediately obtain an estimate on $m_N - m$; this is the reason why we will introduce a convexified version of the measure μ in the next Section 3. To quantify more precisely (2.11) we will use the following result, already proved in [6] for convex V .

Lemma 2.2 Let $\delta > 0$. For $z = E + i\eta$ with $A + \delta < E < B - \delta$ assume that

$$\frac{1}{N^2} k_N(z) \rightarrow 0 \quad (2.12)$$

as $N \rightarrow \infty$ uniformly in $\eta \geq N^{-1+a}$ for some $0 < a < 1$. Then there are constants $c, \kappa > 0$ such that for any $N^{-1+a} \leq \eta \leq \kappa$, $A + \delta < E < B - \delta$,

$$|m_N(z) - m(z)| \leq c \left(\frac{1}{N\eta} + \frac{1}{N^2} k_N(z) \right). \quad (2.13)$$

Proof. First, for technical contour integration reasons, it will be easier to consider the measure (2.4) instead of $\mu^{(N)}$ here. More precisely, define

$$\begin{aligned} m_N^{(\kappa)}(z) &= \mathbb{E}_{\mu^{(N,\kappa)}} \left(\frac{1}{N} \sum_{k=1}^N \frac{1}{z - \lambda_i} \right) = \int_{\mathbb{R}} \frac{1}{z - t} \rho_1^{(N,\kappa)}(t) dt, \\ k_N^{(\kappa)}(z) &= \text{var}_{\mu^{(N,\kappa)}} \left(\sum_{k=1}^N \frac{1}{z - \lambda_i} \right), \\ c_N^{(\kappa)}(z) &= \frac{1}{N^2} k_N^{(\kappa)}(z) + \frac{1}{N} \left(\frac{2}{\beta} - 1 \right) m_N^{(\kappa)'}(z). \end{aligned}$$

Then it is a direct consequence of (2.5) and (2.6) that for any $\kappa > 0$ there is a constant $c > 0$ such that uniformly on $\eta \geq N^{-10}$ (or any power of N),

$$|m_N^{(\kappa)} - m_N| = O(e^{-cN}), \quad |k_N^{(\kappa)} - k_N| = O(e^{-cN}). \quad (2.14)$$

From now, we choose a fixed $\kappa > 0$ such that all the zeros of r are at distance at least 10κ from $[A, B]$ (this is possible because V is regular). Consider the rectangle with vertices $B + 5\kappa + iN^{-10}$, $A - 5\kappa + iN^{-10}$, $A - 5\kappa - iN^{-10}$, $B + 5\kappa - iN^{-10}$, call \mathcal{L} the corresponding clockwise closed contour and \mathcal{L}' the one consisting only in the horizontal pieces, with the same orientation. From (2.10), we obviously have, for $z \notin \mathcal{L}'$,

$$\frac{1}{2\pi i} \int_{\mathcal{L}'} \frac{(m_N(\xi) - m(\xi))^2 + s(\xi)(m_N(\xi) - m(\xi)) + b_N(\xi) - c_N(\xi)}{r(\xi)(z - \xi)} d\xi = 0.$$

Note that the above expression makes sense for large enough N , because then r has no zero on \mathcal{L} . Using (2.14), this implies, for $\eta \geq N^{-1}$,

$$\frac{1}{2\pi i} \int_{\mathcal{L}'} \frac{(m_N^{(\kappa)}(\xi) - m(\xi))^2 + s(\xi)(m_N^{(\kappa)}(\xi) - m(\xi)) + b_N(\xi) - c_N^{(\kappa)}(\xi)}{r(\xi)(z - \xi)} d\xi = O(e^{-cN}).$$

Now, as $\rho_1^{(N,\kappa)}$ and ρ are supported on $[A - \kappa, B + \kappa]$, $m_N^{(\kappa)} - m$ and $c_N^{(\kappa)}$ are uniformly $O(1)$ in the vertical segments of \mathcal{L} . Consequently, from the above equation

$$\frac{1}{2\pi i} \int_{\mathcal{L}} \frac{(m_N^{(\kappa)}(\xi) - m(\xi))^2 + s(\xi)(m_N^{(\kappa)}(\xi) - m(\xi)) + b_N(\xi) - c_N^{(\kappa)}(\xi)}{r(\xi)(z - \xi)} d\xi = O(N^{-10}).$$

As b_N and r are analytic inside \mathcal{L} , for z outside \mathcal{L} we get

$$\frac{1}{2\pi i} \int_{\mathcal{L}} \frac{(m_N^{(\kappa)}(\xi) - m(\xi))^2 + s(\xi)(m_N^{(\kappa)}(\xi) - m(\xi)) - c_N^{(\kappa)}(\xi)}{r(\xi)(z - \xi)} d\xi = O(N^{-10}).$$

Remember we define $f(z) = \sqrt{(A-z)(B-z)}$ uniquely by $f(z) \sim z$ as $z \rightarrow \infty$. Moreover, $|m_N^{(\kappa)} - m|(z) = O(z^{-2})$ as $|z| \rightarrow \infty$ because ρ and $\rho_1^{(N,\kappa)}$ are compactly supported:

$$\begin{aligned} |m_N^{(\kappa)}(z) - m(z)| &= \left| \int_{A-\kappa}^{B+\kappa} \frac{\rho(t) - \rho^{(N,\kappa)}(t)}{z-t} dt \right| \\ &= \left| \int_{A-\kappa}^{B+\kappa} (\rho(t) - \rho^{(N,\kappa)}(t)) \left(\frac{1}{z} + O\left(\frac{1}{z^2}\right) \right) dt \right| = O(z^{-2}). \end{aligned}$$

Consequently, the function $s(m_N^{(\kappa)} - m)/r = -2f(m_N^{(\kappa)} - m)$ is $O(z^{-1})$ as $|z| \rightarrow \infty$. Moreover, it is analytic outside \mathcal{L} , so the Cauchy integral formula yields

$$\frac{1}{2\pi i} \int_{\mathcal{L}} \frac{s(\xi)(m_N^{(\kappa)}(\xi) - m(\xi))}{r(\xi)(z - \xi)} d\xi = -2f(z)(m_N^{(\kappa)} - m)(z),$$

proving

$$-2f(z)(m_N^{(\kappa)}(z) - m(z)) = -\frac{1}{2\pi i} \int_{\mathcal{L}} \frac{(m_N^{(\kappa)}(\xi) - m(\xi))^2 - c_N^{(\kappa)}(\xi)}{r(\xi)(z - \xi)} d\xi + O(N^{-10}). \quad (2.15)$$

Consider now the following rectangular contours, defined by their vertices:

$$\begin{aligned} \mathcal{L}_1 &: B + 3\kappa + i3\kappa, A - 3\kappa + i3\kappa, A - 3\kappa - i3\kappa, B + 3\kappa - i3\kappa, \\ \mathcal{L}_2 &: B + 4\kappa + i4\kappa, A - 4\kappa + i4\kappa, A - 4\kappa - i4\kappa, B + 4\kappa - i4\kappa. \end{aligned} \quad (2.16)$$

In particular, note that all the zeros of r are strictly outside \mathcal{L}_2 . For z inside \mathcal{L}_2 and $\text{Im}(z) \geq N^{-1}$, by the Cauchy formula, equation (2.15) implies that

$$\begin{aligned} &-2s(z)(m_N^{(\kappa)}(z) - m(z)) \\ &= -(m_N^{(\kappa)}(z) - m(z))^2 + c_N^{(\kappa)}(z) - \frac{r(z)}{2\pi i} \int_{\mathcal{L}_2} \frac{(m_N^{(\kappa)}(\xi) - m(\xi))^2 - c_N^{(\kappa)}(\xi)}{r(\xi)(z - \xi)} d\xi + O(N^{-10}). \end{aligned} \quad (2.17)$$

In the above expression, if now z is on \mathcal{L}_1 , $|z - \xi| \geq \kappa$, and on \mathcal{L}_2 $|r|$ is separated away from zero by a positive universal constant. Moreover, $c_N^{(\kappa)}(\xi)$ can be bounded in the following way. For any $\xi \in \mathcal{L}_2$, there is a smooth function g_ξ supported on $[A - 2\kappa, B + 2\kappa]$ which coincides with $\frac{1}{\xi - \lambda_k}$ on $[A - \kappa, B + \kappa]$. Moreover, this choice can be made such that $\|g_\xi\|_\infty, \|g'_\xi\|_\infty, \|g''_\xi\|_\infty$ are uniformly bounded in $\xi \in \mathcal{L}_2$. Then

$$\begin{aligned} \frac{1}{N^2} \left| \text{var}_{\mu^{(N,\kappa)}} \left(\sum_{k=1}^N \frac{1}{\xi - \lambda_k} \right) \right| &= \frac{1}{N^2} \left| \text{var}_{\mu^{(N,\kappa)}} \left(\sum_{k=1}^N g_\xi(\lambda_k) \right) \right| \\ &= \frac{1}{N^2} \left| \text{var}_{\mu^{(N)}} \left(\sum_{k=1}^N g_\xi(\lambda_k) \right) \right| (1 + o(1)), \end{aligned}$$

where the last equality follows from (2.5). Now, from Lemma 2.1, this last variance is uniformly bounded by $c(\log N)^2$, with c uniformly bounded in ξ . This proves that $k_N^{(\kappa)}(\xi)$ is $O((\log N)^2/N^2)$,

uniformly on the contour \mathcal{L}_2 . Moreover, $\frac{1}{N}m_N^{(\kappa)'} = O(N^{-1})$, so finally $c_N^{(\kappa)}(\xi)$ is uniformly $O(N^{-1})$ on \mathcal{L}_2 and (2.17) implies

$$-2s(z)(m_N^{(\kappa)}(z) - m(z)) = -(m_N^{(\kappa)}(z) - m(z))^2(z) + O\left(\sup_{\mathcal{L}_2} |m_N^{(\kappa)} - m|^2\right) + O(N^{-1}).$$

Moreover, from the maximum principle for analytic functions, $\sup_{\mathcal{L}_2} |m_N^{(\kappa)} - m| \leq \sup_{\mathcal{L}_1} |m_N^{(\kappa)} - m|$, so the previous equation implies

$$\sup_{\mathcal{L}_1} |m_N^{(\kappa)} - m| = O\left(\sup_{\mathcal{L}_1} |m_N^{(\kappa)} - m|^2 + \frac{1}{N}\right).$$

We know that $\rho_1^{(N)}(s)ds$ converges weakly to $\rho(s)ds$ (see [2]), so by (2.5) and (2.6) $\rho_1^{(N,\kappa)}(s)ds$ converges weakly to $\rho(s)ds$. On \mathcal{L}_1 , z is at distance at least κ from the support of both $\rho_1^{(N,\kappa)}(s)ds$ and $\rho(s)ds$ so, on \mathcal{L}_1 , $m_N^{(\kappa)} - m$ converges uniformly to 0. Together with the above equation, this implies that

$$\sup_{\mathcal{L}_1} |m_N^{(\kappa)} - m| = O\left(\frac{1}{N}\right).$$

By the maximum principle the same estimate holds outside \mathcal{L}_1 , in particular on \mathcal{L}_2 , so equation (2.17) implies that for z inside \mathcal{L}_1

$$-2s(z)(m_N^{(\kappa)}(z) - m(z)) = -(m_N^{(\kappa)}(z) - m(z))^2 + c_N^{(\kappa)}(z) + O\left(\frac{1}{N}\right). \quad (2.18)$$

Moreover,

$$\begin{aligned} \frac{1}{N}|m_N^{(\kappa)'}(z)| &= \frac{1}{N^2} \left| \mathbb{E}_{\mu^{(N,\kappa)}} \sum_j \frac{1}{(z - \lambda_j)^2} \right| \\ &\leq \frac{1}{N\eta} \operatorname{Im} m_N^{(\kappa)}(z) \leq \frac{1}{N\eta} |m_N^{(\kappa)}(z) - m(z)| + \frac{1}{N\eta} |\operatorname{Im} m(z)| \leq \frac{1}{N\eta} |m_N^{(\kappa)}(z) - m(z)| + \frac{c}{N\eta} \end{aligned} \quad (2.19)$$

for some constant c . We used the well-known fact that $\operatorname{Im} m$ is uniformly bounded on the upper half plane (this follows for example from properties of the Cauchy operator, see p 183 in [9]). On the set $A + \delta < E < B - \delta$ and $|\eta| < \kappa$, we have $\inf |s| > 0$. Therefore (2.18) takes the form

$$\left(1 + O\left(\frac{1}{N\eta}\right)\right) (m_N^{(\kappa)}(z) - m(z)) = O\left(|m_N^{(\kappa)}(z) - m(z)|^2 + \frac{1}{N^2}k_N^{(\kappa)}(z) + \frac{1}{N\eta}\right). \quad (2.20)$$

From the hypothesis (2.12), if $N^{-1+a} \leq \eta \leq \kappa$ and $A + \delta < E < B - \delta$, then

$$|m_N^{(\kappa)} - m| \leq c|m_N^{(\kappa)} - m|^2 + \varepsilon_N, \quad (2.21)$$

for some $c > 0$ and $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. For large N , (2.21) implies that $|m_N^{(\kappa)} - m| \leq 2\varepsilon_N$ or $|m_N^{(\kappa)} - m| \geq 1/c - 2\varepsilon_N$. Together with $|m_N^{(\kappa)} - m|(E + i\kappa) \rightarrow 0$ and the continuity of $|m_N^{(\kappa)} - m|$ in the upper half plane, this implies that $|m_N^{(\kappa)} - m| \leq 2\varepsilon_N$ and therefore $|m_N^{(\kappa)} - m| \rightarrow 0$ uniformly

on $N^{-1+a} \leq \eta \leq \kappa$, $A + \delta < E < B - \delta$. Consequently, using (2.20), this proves that there is a constant $c > 0$ such that for any $\eta \geq N^{-1+a}$, $A + \delta < E < B - \delta$,

$$|m_N^{(\kappa)}(z) - m(z)| \leq c \left(\frac{1}{N\eta} + \frac{1}{N^2} k_N^{(\kappa)}(z) \right).$$

The same conclusion remains when substituting $m_N^{(\kappa)}$ (resp. $k_N^{(\kappa)}$) by m_N (resp. k_N) thanks to (2.5) and (2.6). \square

To prove rigidity results for μ , the above Lemma 2.2 will be combined with the following Helffer-Sjöstrand estimate, already proved in the following form in [6].

Lemma 2.3 *Let $\delta < (B-A)/2$ and $E \in [A+\delta, B-\delta]$ and $0 < \eta < \delta/2$. Define a function $f = f_{E,\eta}: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = 1$ for $x \in (-\infty, E - \eta]$, $f(x)$ vanishes for $x \in [E + \eta, \infty)$, moreover $|f'(x)| \leq c\eta^{-1}$ and $|f''(x)| \leq c\eta^{-2}$, for some constant c . Let $\tilde{\rho}$ be an arbitrary signed measure and let $S(z) = \int (z-x)^{-1} \tilde{\rho}(x) dx$ be its Stieltjes transform. Assume that, for any $x \in [A + \delta/2, B - \delta/2]$,*

$$|S(x + iy)| \leq \frac{U}{Ny} \text{ for } \eta < y < 1, \text{ and } |\operatorname{Im} S(x + iy)| \leq \frac{U}{Ny} \text{ for } 0 < y < \eta. \quad (2.22)$$

Assume moreover that $\int_{\mathbb{R}} \tilde{\rho}(\lambda) d\lambda = 0$ and that there is a real constant \mathcal{T} such that

$$\int_{[-\mathcal{T}, \mathcal{T}]^c} |\lambda \tilde{\rho}(\lambda)| d\lambda \leq \frac{U}{N}. \quad (2.23)$$

Then for some constant $C > 0$, independent of N and $E \in [A + \delta, B - \delta]$, we have

$$\left| \int f_E(\lambda) \tilde{\rho}(\lambda) d\lambda \right| \leq \frac{CU |\log \eta|}{N}.$$

3 Convexification

3.1 Outline of the main ideas

The Hamiltonian $\mathcal{H} = \mathcal{H}_N$ of the measure $\mu \sim \exp(-\beta N\mathcal{H})$ is given by

$$\mathcal{H} = \frac{1}{2} \sum_{k=1}^N V(\lambda_k) - \frac{1}{N} \sum_{1 \leq i < j \leq N} \log(\lambda_j - \lambda_i).$$

\mathcal{H} is not convex, but its Hessian is bounded from below, $\nabla^2 \mathcal{H} \geq -W$. We will modify this Hamiltonian by an additional term

$$\tilde{\mathcal{H}} := \mathcal{H} + M \sum_{\alpha=1}^{\ell} X_{\alpha}^2, \quad X_{\alpha} = N^{-1/2} \sum_{j=1}^N (g_{\alpha}(\lambda_j) - g_{\alpha}(\tilde{\gamma}_j)), \quad (3.1)$$

where the real functions g_{α} , $\alpha = 1, 2, \dots, \ell$, will be determined later and will be independent of N . Here we denoted by $\tilde{\gamma}_j$ a slightly modified version of the classical location of the points, defined by the relation

$$\int_A^{\tilde{\gamma}_j} \rho(s) ds = \frac{j - \frac{1}{2}}{N}, \quad j = 1, 2, \dots, N. \quad (3.2)$$

Compared with γ_j defined in (1.6), there is a small shift in the definition which makes a technical step (Lemma 3.3) easier in this section. In all estimates involving γ_j this small shift plays no role since $\max_j |\gamma_j - \tilde{\gamma}_j| \leq CN^{-2/3}$. In particular the crude large deviation bound (1.7) holds for $\tilde{\gamma}$'s as well:

$$\mathbb{P}_\mu(\exists k \in \llbracket 1, N \rrbracket \mid |\lambda_k - \tilde{\gamma}_k| \geq \varepsilon) \leq c_1 e^{-c_2 N}. \quad (3.3)$$

The $N^{-1/2}$ normalization in the definition of X_α is chosen such that the vector

$$\mathbf{G}_\alpha := N^{-1/2}(g'_\alpha(\tilde{\gamma}_1), g'_\alpha(\tilde{\gamma}_2), \dots, g'_\alpha(\tilde{\gamma}_N)) \in \mathbb{R}^N$$

is ℓ^2 -normalized.

Define the random variables

$$\Delta := \max \left(\frac{1}{N} \sum_j |\lambda_j - \tilde{\gamma}_j|, \frac{1}{N} \sum_j (\lambda_j - \tilde{\gamma}_j)^2 \right) \leq \Delta^{(\delta)} := \delta + \frac{1}{N\delta} \sum_j (\lambda_j - \tilde{\gamma}_j)^2$$

for any $0 < \delta < 1$. Clearly

$$|X_\alpha| \leq N^{1/2} \|g'_\alpha\|_\infty \Delta.$$

We then have, for any vector $\mathbf{v} \in \mathbb{R}^N$, that

$$\begin{aligned} \langle \mathbf{v}, (\nabla^2 \tilde{\mathcal{H}}) \mathbf{v} \rangle &= \frac{1}{N} \sum_{i < j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} + \frac{1}{2} \sum_j V''(\lambda_j) v_j^2 \\ &\quad + 2M \sum_{\alpha=1}^\ell \left[\left(\frac{1}{\sqrt{N}} \sum_j g'_\alpha(\lambda_j) v_j \right)^2 + X_\alpha \sum_j \frac{1}{\sqrt{N}} g''_\alpha(\lambda_j) v_j^2 \right] \\ &\geq \frac{1}{N} \sum_{i < j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} + M \sum_{\alpha=1}^\ell |\langle \mathbf{G}_\alpha, \mathbf{v} \rangle|^2 \\ &\quad - \left[W + 2\Delta M \sum_{\alpha=1}^\ell (\|g''_\alpha\|_\infty^2 + \|g'_\alpha\|_\infty \|g''_\alpha\|_\infty) \right] \|\mathbf{v}\|^2 \end{aligned} \quad (3.4)$$

where we used a simple Schwarz inequality

$$\begin{aligned} 2 \left(\sum_j g'_\alpha(\lambda_j) v_j \right)^2 &\geq \left(\sum_j g'_\alpha(\tilde{\gamma}_j) v_j \right)^2 - 2 \left(\sum_j [g'_\alpha(\lambda_j) - g'_\alpha(\tilde{\gamma}_j)] v_j \right)^2 \\ &\geq \left(\sum_j g'_\alpha(\tilde{\gamma}_j) v_j \right)^2 - 2 \sum_j [g'_\alpha(\lambda_j) - g'_\alpha(\tilde{\gamma}_j)]^2 \|\mathbf{v}\|^2 \end{aligned}$$

in the last step.

We will define below a nonnegative symmetric operator \mathcal{Q} on \mathbb{C}^N via a quadratic form

$$\langle \mathbf{v}, \mathcal{Q} \mathbf{v} \rangle = \sum_{i,j=1}^N Q_{ij} (v_i - v_j)^2 \quad (3.5)$$

such that for typical point configuration $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)$ we have

$$\frac{1}{N} \sum_{i < j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} \geq \sum_{i,j=1}^N Q_{ij} (v_i - v_j)^2. \quad (3.6)$$

In our applications, we will then choose ℓ to be a large but N -independent number, we will let \mathbf{G}_α , $\alpha = 1, 2, \dots, \ell$ be the eigenfunctions corresponding to the lowest ℓ eigenvalues $\mu_1 \leq \mu_2 \leq \dots \leq \mu_\ell$ of the nonnegative operator \mathcal{Q} . Thus we will have the operator inequality

$$\mathcal{Q} + M \sum_{\alpha=1}^{\ell} |\mathbf{G}_\alpha\rangle\langle\mathbf{G}_\alpha| \geq \min\{M, \mu_{\ell+1}\}. \quad (3.7)$$

In Section 3.2 we will show that for ℓ sufficiently large, independent of N , we have $\mu_{\ell+1} > M$. Setting

$$C(\ell) := 2M \sum_{\alpha=1}^{\ell} (\|g''_\alpha\|_\infty^2 + \|g'_\alpha\|_\infty \|g''_\alpha\|_\infty),$$

we will obtain from (3.4) that

$$\langle \mathbf{v}, (\nabla^2 \tilde{\mathcal{H}}) \mathbf{v} \rangle \geq \left(M - W - C(\ell) \Delta^{(\delta)} \right) \|\mathbf{v}\|^2 \geq 0 \quad (3.8)$$

as long as $\Delta^{(\delta)} \leq 1/C(\ell)$. From now, we choose $\delta = 1/(2C(\ell))$ and as $N \rightarrow \infty$, we have $\Delta^{(\delta)} \leq 1/C(\ell)$ with very high probability, thanks to the large deviation estimates (3.3). To summarize, for given W we first chose M large enough, then ℓ so that (3.7) and $\mu_{\ell+1} > M$ hold, and then $\delta = 1/(2C(\ell))$, and we obtain $\langle \mathbf{v}, (\nabla^2 \tilde{\mathcal{H}}) \mathbf{v} \rangle \geq 0$.

To prove that $\mu_{\ell+1} \geq M$, we only need to estimate the low lying eigenvalues of \mathcal{Q} and we need to understand the low lying eigenfunctions \mathbf{G}_α . Since the only requirement for $Q_{i,j}$ is to satisfy the bound (3.6), we have a substantial freedom in choosing $Q_{i,j}$ conveniently. There are many ways to choose $Q_{i,j}$; we will give one possible approach that relies on enlarging the space by a reflection principle in the next section. Roughly speaking, we will construct an operator \mathcal{R} with periodic boundary conditions on the set consisting of the original set and its ‘‘reflection’’. We then choose \mathcal{Q} to be the restriction of \mathcal{R} to the symmetric (under the reflection) sector. The operator \mathcal{R} is translation invariant, hence it can be diagonalized via Fourier transform and the eigenfunctions are explicit. The reader may skip the next section on first reading as it contains fairly elementary arguments that are independent of the rest of the paper.

3.2 Slow modes analysis

Let $I := \llbracket 1, N \rrbracket$ be the index set of the vectors \mathbf{v} . The original operator \mathcal{Q} is defined on the space $\ell^2(I)$. We enlarge this space to $\ell^2(\tilde{I})$, where $\tilde{I} := \llbracket -N + 1, N \rrbracket$. We extend any vector $\mathbf{v} \in \ell^2(I)$ by reflection to a vector $\tilde{\mathbf{v}} \in \ell^2(\tilde{I})$ as follows

$$\tilde{v}_j = v_j, \quad j = 1, 2, \dots, N; \quad (3.9)$$

$$\tilde{v}_j = v_{1-j}, \quad j = 0, \dots, -N + 1. \quad (3.10)$$

We will often view the set \tilde{I} modulo $2N$ periodic, i.e. we consider it as $2N$ points on a circle and identify $-N$ with N . We can thus also view $\ell^2(\tilde{I})$ as the space of vectors with periodic boundary condition $\tilde{v}_{-N} = \tilde{v}_N$. The algebraic operations on the indices will be considered modulo $2N$.

We consider the natural translation invariant distance on \tilde{I} . Define the function $m(n)$ for $n \in \mathbb{Z}$ such that $m(n) \in \llbracket -N+1, N \rrbracket$ and $m(n) \equiv n \pmod{2N}$. Then the distance between $k, \ell \in \tilde{I}$ is defined as $d(k, \ell) := |m(k) - m(\ell)|$ which ranges from 0 to N .

Lemma 3.1 *Let $\varepsilon > 0$ be sufficiently small, depending only on V . Define*

$$R_{k,\ell}^{(\varepsilon)} = R_{k,\ell} := \frac{1}{N} \frac{\varepsilon^{2/3}}{\frac{d(k,\ell)^2}{N^2} + \varepsilon^2}, \quad k, \ell \in \tilde{I} = \llbracket -N+1, N \rrbracket, \quad (3.11)$$

and

$$Q_{i,j}^{(\varepsilon)} = Q_{i,j} := R_{i,j} + R_{1-i,j} + R_{i,1-j} + R_{1-i,1-j}, \quad i, j \in I = \llbracket 1, N \rrbracket. \quad (3.12)$$

Then there is a constant $c_1 > 0$ depending only on V such that for any $\varepsilon > 0$ there is a constant $c_2 > 0$ (depending on V and ε) such that for any N and $i, j \in \llbracket 1, N \rrbracket$

$$\mathbb{P}_\mu \left(\frac{1}{N} \frac{1}{(\lambda_i - \lambda_j)^2} \leq c_1 Q_{i,j} \right) \leq e^{-c_2 N}.$$

The relation between $Q_{i,j}$ and $R_{i,j}$ is dictated by the requirement that

$$\langle \mathbf{v}, \mathcal{Q}\mathbf{v} \rangle_{\ell^2(I)} = \langle \tilde{\mathbf{v}}, \mathcal{R}\tilde{\mathbf{v}} \rangle_{\ell^2(\tilde{I})} := \sum_{i,j \in \tilde{I}} R_{i,j} (\tilde{v}_i - \tilde{v}_j)^2, \quad \forall \mathbf{v} \in \ell^2(I),$$

which can be easily checked from (3.9).

Proof. Recall that $[A, B]$ is the support of ρ , $\rho > 0$ on (A, B) and ρ has a square-root singularity at the two endpoints, i.e. it vanishes as $\rho(x) \sim s_A \sqrt{x - A}$ as $x \rightarrow A^+$ and $\rho(x) \sim s_B \sqrt{B - x}$ as $x \rightarrow B^-$ with some positive s_A, s_B .

From the large deviations of the extreme eigenvalues (included in (1.7)), we know that for any $\kappa > 0$ there is a $c(\kappa) > 0$ such that

$$\mathbb{P}_\mu \left(\lambda_1 \leq A - \kappa \right) + \mathbb{P}_\mu \left(\lambda_N \geq B + \kappa \right) \leq e^{-c(\kappa)N}. \quad (3.13)$$

Fix a positive number $s < \min(s_A, s_B)$. Then there is a $\kappa_0 > 0$, depending only on V , such that

$$\rho(x) \geq s\sqrt{x - A} \cdot \mathbb{1}_{x \in [A, A + \kappa_0]} + s\sqrt{\kappa_0} \cdot \mathbb{1}_{x \in [A + \kappa_0, B - \kappa_0]} + s\sqrt{B - x} \cdot \mathbb{1}_{x \in [B - \kappa_0, A]}. \quad (3.14)$$

Let $\varepsilon \leq c\kappa_0^{3/2}$ with a small positive constant c . Suppose that $k \leq N/2$; if k is near the upper edge, the argument is similar. Since

$$\int_{-\infty}^{\tilde{\gamma}_k} \rho = \frac{k - \frac{1}{2}}{N} = \int_{-\infty}^{\lambda_k} \frac{1}{N} \sum_m \delta_{\lambda_m} - \frac{1}{2N},$$

from the first relation we get

$$c(k/N)^{2/3} \leq \tilde{\gamma}_k - A \leq C(k/N)^{2/3} \quad (3.15)$$

with some positive constants c, C , depending only on V . Subtracting the first and second relations and using (3.3), we obtain that for any fixed K

$$\left| \int_{\tilde{\gamma}_k}^{\lambda_k} \rho \right| \leq \frac{\varepsilon}{K} \quad (3.16)$$

apart from an event of exponentially small probability (i.e. of type $\exp(-c(\varepsilon/K)N)$).

Additionally, assume now that $k \geq N\varepsilon$. Under (3.16) we easily see that $\lambda_k \in (\tilde{\gamma}_{k/2}, \tilde{\gamma}_{3k/2})$, since both $\int_{\tilde{\gamma}_{k/2}}^{\tilde{\gamma}_k} \rho$ and $\int_{\tilde{\gamma}_k}^{\tilde{\gamma}_{3k/2}} \rho$ are of the order k/N which is larger than ε/K if K is large enough (depending only on V). Then (3.14) and (3.16) imply

$$|\lambda_k - \tilde{\gamma}_k| \leq C\varepsilon(\tilde{\gamma}_k - A)^{-1/2} \leq C\varepsilon(k/N)^{-1/3}, \quad N\varepsilon \leq k \leq N/2$$

with exponentially high probability and with a constant C depending only on V .

Now we consider the $k \leq N\varepsilon$ case. Using (3.13) with $\kappa = \varepsilon^{2/3}$ and (3.15), we have (apart from an event of exponentially small probability)

$$\tilde{\gamma}_k - \lambda_k \leq C(k/N)^{2/3} + \kappa \leq C\varepsilon^{2/3}.$$

Finally, still when $k \leq N\varepsilon$, i.e. $\tilde{\gamma}_k \leq A + C\varepsilon^{2/3}$ then (3.16) implies that $\lambda_k \leq A + C_1\varepsilon^{2/3}$ with a large C_1 , i.e.

$$\lambda_k - \tilde{\gamma}_k \leq C\varepsilon^{2/3}, \quad k \leq N\varepsilon,$$

still apart from an event of exponentially small probability. Summarizing all cases, we obtain that

$$|\lambda_k - \tilde{\gamma}_k| \leq \frac{C\varepsilon}{(k/N)^{1/3} + \varepsilon^{1/3}}, \quad k \leq N/2. \quad (3.17)$$

holds with overwhelming probability.

Now let $|i - j| \geq N\varepsilon$, then

$$\frac{|i - j|}{N} = \left| \int_{\tilde{\gamma}_i}^{\tilde{\gamma}_j} \rho \right| \geq c|\tilde{\gamma}_i - \tilde{\gamma}_j|\varepsilon^{1/3}$$

since either i or j is larger than $N\varepsilon$ and smaller than $N(1 - \varepsilon)$, say $N\varepsilon \leq i \leq N(1 - \varepsilon)$, and then ρ is at least of order $\varepsilon^{1/3}$ in the neighborhood of $\tilde{\gamma}_i$. If $|i - j| \leq N\varepsilon$, then we have the trivial bound $|\tilde{\gamma}_i - \tilde{\gamma}_j| \leq C\varepsilon^{2/3}$. Combining these,

$$|\tilde{\gamma}_i - \tilde{\gamma}_j| \leq \frac{C|i - j|}{N\varepsilon^{1/3}} + C\varepsilon^{2/3}$$

holds for any i, j . Furthermore, clearly $|\lambda_i - \tilde{\gamma}_i| \leq C\varepsilon^{2/3}$ from (3.17), so we have proved that

$$|\lambda_i - \lambda_j| \leq \frac{C|i - j|}{N\varepsilon^{1/3}} + C\varepsilon^{2/3}$$

with overwhelming probability and for any i, j . In other words, there is a constant C (depending only on V) such that for any sufficiently small ε and for some $c(\varepsilon) > 0$ we have for any N and $i, j \in \llbracket 1, N \rrbracket$

$$\mathbb{P} \left(\frac{1}{(\lambda_i - \lambda_j)^2} < \frac{C\varepsilon^{2/3}}{\varepsilon^2 + \frac{|i-j|^2}{N^2}} \right) < e^{-c(\varepsilon)N}. \quad (3.18)$$

The proof of Lemma 3.1 will therefore be complete if we can prove that

$$|i - j| \leq \min \{d(i, j), d(1 - i, j), d(i, 1 - j), d(1 - i, 1 - j)\}, \quad i, j \in I.$$

Note that we can assume $i > j$ and then $|i - j| = |m(i - j)| = d(i, j)$ is obvious. Moreover, as d is symmetric to the reflection $i \rightarrow 1 - i$ on \tilde{I} , i.e. $d(k, \ell) = d(1 - k, 1 - \ell)$, we just need to prove that $|i - j| \leq d(1 - i, j)$. If $i + j \leq N + 1$, then $d(1 - i, j) = i + j - 1 > |i - j|$. If $N + 2 \leq i + j \leq 2N$, then $d(1 - i, j) = |m(i + j - 1)| = |i + j - 1 - 2N| = 2N + 1 - i - j > i - j$ because $i \leq N$, completing the proof. \square

We use the matrix Q in the previous lemma instead of bounds of type (3.18) because it is related to R , a circulant matrix, allowing to derive its eigenvalues and eigenvectors in an explicit way.

Lemma 3.2 *Let $R = R^{(\varepsilon)}$ be the matrix $(R_{i,j})_{(i,j) \in \tilde{I}^2}$, where the matrix elements $R_{i,j}$ are defined in (3.11). Then the eigenvalues ν_1, \dots, ν_{2N} of R are*

$$\nu_k = \sum_{j=0}^{2N-1} e^{i2\pi j \frac{k}{2N}} R_{0,j}$$

and the associated normalized eigenvector is $u_k^* = (2N)^{-1/2} (e^{i2\pi j \frac{k}{2N}})_{j=-N+1, \dots, N}$.

In particular, for any given $W > 0$ there is a sufficiently small ε such that for large enough N we have $\nu_{2N} > W$. Moreover, for any given $\varepsilon > 0$ and $s > 0$ there is some $a > 0$ depending only on ε and s such that for any N

$$\{k : |\nu_k| > s\} \subset \llbracket 1, a \rrbracket \cup \llbracket 2N + 1 - a, 2N \rrbracket.$$

We remark that the matrix R defines a symmetric operator $\mathbf{w} \rightarrow R\mathbf{w}$ and a quadratic form $\langle \mathbf{w}, R\mathbf{w} \rangle = \sum R_{i,j} w_i w_j$ in $\ell^2(\tilde{I})$. It is related to the quadratic form \mathcal{R} via

$$\langle \mathbf{w}, \mathcal{R}\mathbf{w} \rangle = \sum_{i,j} R_{i,j} (w_i^2 + w_j^2) - 2 \sum_{i,j} R_{i,j} w_i w_j = 2\nu_{2N} \|\mathbf{w}\|^2 - 2\langle \mathbf{w}, R\mathbf{w} \rangle, \quad (3.19)$$

i.e. we have $\mathcal{R} = 2\nu_{2N} - 2R$ on $\ell^2(\tilde{I})$.

Proof. The first assertions, about the eigenvalues and eigenvectors, is a general fact about circulant matrices and can be obtained by Fourier transform in $\{0, \frac{1}{2N}, \dots, \frac{2N-1}{2N}\}$.

Concerning the distribution of eigenvalues, note that

$$\nu_{2N} = \frac{1}{N} \sum_{j=1}^{2N-1} \frac{\varepsilon^{2/3}}{\frac{m(j)^2}{N^2} + \varepsilon^2} \xrightarrow{N \rightarrow \infty} \varepsilon^{2/3} \int_{-1}^1 \frac{dx}{x^2 + \varepsilon^2} = \varepsilon^{-1/3} \int_{-1/\varepsilon}^{1/\varepsilon} \frac{dx}{x^2 + 1}.$$

We therefore have, for sufficiently small $\varepsilon > 0$, $\nu_{2N} > W$ for large enough N . We now write

$$a_j = \sum_{\ell=0}^j e^{i2\pi \ell \frac{k}{2N}}, \quad b_j = \frac{1}{N} \frac{\varepsilon^{2/3}}{\frac{m(j)^2}{N^2} + \varepsilon^2}, \quad 0 \leq j \leq 2N - 1,$$

with the convention $b_{2N} = 0$. Then note that $|a_j| < 2|1 - e^{i2\pi \frac{k}{2N}}|^{-1}$, and write $\nu_k = \sum_{j=0}^{2N-1} a_j (b_j - b_{j+1})$ to obtain

$$|\nu_k| \leq 4\varepsilon^{2/3} \frac{1}{N|1 - e^{i2\pi \frac{k}{2N}}|} \sum_{j=0}^N \left| \frac{1}{\varepsilon^2 + \left(\frac{j}{N}\right)^2} - \frac{1}{\varepsilon^2 + \left(\frac{j+1}{N}\right)^2} \right| \leq \frac{c(\varepsilon)}{N|1 - e^{i2\pi \frac{k}{2N}}|}.$$

Consequently, if $|\nu_k| > s$ then $|1 - e^{i2\pi \frac{k}{2N}}| < c(\varepsilon, s)/N$, which only happens for indices k whose distance to 0 or $2N$ is uniformly bounded. \square

Lemma 3.3 *For a fixed $\alpha \in \mathbb{N}$, there exists an N -independent function g_α with $\|g_\alpha\|_\infty + \|g'_\alpha\|_\infty + \|g''_\alpha\|_\infty < \infty$ such that*

$$g'_\alpha(\tilde{\gamma}_k) = \sqrt{2} \cos\left(2\pi\left(k - \frac{1}{2}\right)\frac{\alpha}{2N}\right).$$

Define

$$\mathbf{G}_\alpha = N^{-1/2}(g'_\alpha(\tilde{\gamma}_1), \dots, g'_\alpha(\tilde{\gamma}_N)) = \sqrt{2}N^{-1/2} \left(\cos\left(\pi(2j-1)\frac{\alpha}{2N}\right) \right)_{j=1, \dots, N},$$

it is easy to see that $\|\mathbf{G}_\alpha\| = 1$. For any $M > 0$ there is $\varepsilon > 0$ and $\ell > 0$ such that, for large enough N we have, as an inequality between positive operators on $\ell^2(I)$,

$$\mathcal{Q} + M \sum_{\alpha=0}^{\ell} |\mathbf{G}_\alpha\rangle\langle \mathbf{G}_\alpha| \geq M,$$

where $\mathcal{Q} = \mathcal{Q}^{(\varepsilon)}$ was defined in (3.5) with coefficients $Q_{i,j} = Q_{i,j}^{(\varepsilon)}$ defined in (3.12).

Proof. The existence of g_α follows easily from the fact that the density ρ has a square-root singularity near the edges, see (1.4). In fact, we can choose the functions g_α such that $g'_\alpha(x) = \sqrt{2} \cos\left(\frac{2}{3}\pi s_A \alpha (x-A)_+^{3/2}\right)$ for x near A , and a similar relation holds near the upper edge. As $\langle \tilde{\mathbf{v}}, \mathcal{R}\tilde{\mathbf{v}} \rangle = \langle \mathbf{v}, \mathcal{Q}\mathbf{v} \rangle$, $\|\tilde{\mathbf{v}}\|^2 = 2\|\mathbf{v}\|^2$ and

$$|\langle \mathbf{v}, \mathbf{G}_\alpha \rangle|^2 = |\langle \tilde{\mathbf{v}}, u_\alpha \rangle|^2 = |\langle \tilde{\mathbf{v}}, u_{2N-\alpha} \rangle|^2,$$

we just need to prove that the operator inequality

$$2\mathcal{R} + M \sum_{k \in \llbracket 0, \ell \rrbracket \cup \llbracket 2N-\ell, 2N \rrbracket} |u_k\rangle\langle u_k| \geq M$$

holds in $\ell^2(\tilde{I})$ for some fixed constant ℓ and for large enough N . This is equivalent to the statement

$$R + \frac{M}{4} \leq \nu_{2N} + \frac{M}{4} \sum_{k \in \llbracket 0, \ell \rrbracket \cup \llbracket 2N-\ell, 2N \rrbracket} |u_k\rangle\langle u_k|, \quad (3.20)$$

since $\mathcal{R} = 2\nu_{2N} - 2R$, see (3.19). Recalling that ν_k are the eigenvalues of $R = R^{(\varepsilon)}$, we need to check that for some fixed $\varepsilon > 0$ and ℓ we have

$$\nu_k + \frac{M}{4} \leq \nu_{2N} + \frac{M}{4} \mathbf{1}_{k \in \llbracket 0, \ell \rrbracket \cup \llbracket 2N-\ell, 2N \rrbracket}$$

for any k . Since ν_{2N} is the top eigenvalue, this inequality is obvious if $k \in \llbracket 0, \ell \rrbracket \cup \llbracket 2N-\ell, 2N \rrbracket$. Moreover, Lemma 3.2 proves the existence of some fixed $\varepsilon > 0$ and ℓ , such that $\nu_{2N} > M$ and $\{k : |\nu_k| \leq \frac{3}{4}M\} \subset \llbracket 0, \ell \rrbracket \cup \llbracket 2N-\ell, 2N \rrbracket$ hold for large enough N (depending on M as well as ε and ℓ). This concludes the proof. \square

3.3 The locally constrained measures

In this section some arbitrary $\varepsilon, \alpha > 0$ are fixed. Let θ be a continuous nonnegative function with $\theta = 0$ on $[-1, 1]$ and $\theta'' \geq 1$ for $|x| > 1$. We can take for example $\theta(x) = (x-1)^2 \mathbf{1}_{x>1} + (x+1)^2 \mathbf{1}_{x<-1}$ in the following.

Definition 3.4 For any $s, \ell > 0$, define the probability measure

$$d\nu^{(s,\ell)} = e^{-\beta N \mathcal{H}_\nu} := \frac{1}{Z^{(s,\ell)}} e^{-\beta N \psi^{(s)} - \beta N \sum_{i,j} \psi_{i,j} - \beta N (W+1) \sum_{\alpha=0}^{\ell} X_\alpha^2} d\mu, \quad (3.21)$$

where

- the function g_α was defined in Lemma 3.3;
- $X_\alpha = N^{-1/2} \sum_j (g_\alpha(\lambda_j) - g_\alpha(\tilde{\gamma}_j))$;
- $\psi^{(s)}(\lambda) = N\theta\left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2\right)$;
- $\psi_{i,j}(\lambda) = \frac{1}{N} \theta\left(\sqrt{c_1 N Q_{i,j}} (\lambda_i - \lambda_j)\right)$, where c_1 was defined in Lemma 3.1.

Lemma 3.5 There are constants $c, \ell, s > 0$ depending only on V , such that for N large enough $\nu^{(s,\ell)}$ satisfies, for any $\mathbf{v} \in \mathbb{R}^N$,

$$\langle \mathbf{v}, (\nabla^2 \mathcal{H}_\nu) \mathbf{v} \rangle \geq c \|\mathbf{v}\|^2.$$

Proof. Using the notation (3.1), we have $\mathcal{H}_\nu = \tilde{\mathcal{H}} + \psi^{(s)} + \sum_{i,j} \psi_{i,j}$ up to an additive constant, so

$$\begin{aligned} \langle \mathbf{v}, (\nabla^2 \mathcal{H}_\nu) \mathbf{v} \rangle &= \langle \mathbf{v}, (\nabla^2 \tilde{\mathcal{H}}) \mathbf{v} \rangle + c_1 \sum_{i,j} Q_{i,j} \theta''\left(\sqrt{c_1 N Q_{i,j}} (\lambda_i - \lambda_j)\right) (v_i - v_j)^2 \\ &\quad + \frac{(2s)^2}{N} \theta''\left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2\right) \left(\sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i) v_i\right)^2 + 2s \theta'\left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2\right) \|\mathbf{v}\|^2. \end{aligned} \quad (3.22)$$

We now use (3.4) to get a lower bound for $\langle \mathbf{v}, (\nabla^2 \tilde{\mathcal{H}}) \mathbf{v} \rangle$. Note that the second θ'' term is positive and that for $x > 0$, $\theta'(x) \geq 2(x-1) \mathbf{1}_{x>1}$; we therefore get the following lower estimate of (3.22):

$$\begin{aligned} \frac{1}{N} \sum_{i,j} \left(\frac{1}{(\lambda_i - \lambda_j)^2} + c_1 N Q_{i,j} \mathbf{1}\left(\frac{1}{(\lambda_i - \lambda_j)^2} < c_1 N Q_{i,j}\right) \right) (v_i - v_j)^2 + (W+1) \sum_{\alpha=1}^{\ell} |\langle \mathbf{G}_\alpha, \mathbf{v} \rangle|^2 \\ - \left(W + C(\ell) \Delta^{(\delta)} \right) \|\mathbf{v}\|^2 + 4s(s\Delta - 1) \mathbf{1}_{s\Delta - 1 > 0} \|\mathbf{v}\|^2, \end{aligned}$$

which is greater than

$$\left\langle \mathbf{v}, (c_1 \mathcal{Q} + (W+1) \sum_{\alpha=1}^{\ell} |\mathbf{G}_\alpha\rangle \langle \mathbf{G}_\alpha| - (W+1) \mathbf{v}) \mathbf{v} \right\rangle + (1 - C(\ell) \Delta^{(\delta)}) \|\mathbf{v}\|^2 + 4s(s\Delta - 1) \mathbf{1}_{s\Delta - 1 > 0} \|\mathbf{v}\|^2.$$

Choosing $M = (W+1)/c_1$ in Lemma 3.3, for ε small enough and ℓ large enough the above scalar product term is positive for any \mathbf{v} and large enough N (note that c_1 does not depend on ε). For this choice of ℓ , taking $s = C(\ell)$ makes the other terms all together positive (without loss of generality we can assume $C(\ell) > 1/4$), concluding the proof. \square

From now, we abbreviate ν for $\nu^{(s,\ell)}$, where s and ℓ are fixed such that the conclusion of Lemma 3.5 holds.

3.4 Equivalence of the measures ν and μ

We say that a sequence of events $(A_N)_{N \geq 1}$ is exponentially small for a sequence of probability measures $(m_N)_{N \geq 1}$ if there are constants $\delta, c_1, c_2 > 0$ such that for any N we have

$$m_N(A_N) \leq c_1 e^{-c_2 N^\delta}.$$

Lemma 3.6 *For any $s, \ell > 0$, the measures $(\mu^{(N)})_{N \geq 1}$ and $(\nu^{(s, \ell, N)})_{N \geq 1}$ have the same exponentially small events.*

Proof. First note that $\mathcal{H}_\nu \geq \mathcal{H}_\mu$, so $Z_\nu \leq Z_\mu$. We claim that the following inequality holds:

$$\log Z_\mu \leq \log Z_\nu + \mathbb{E}_\mu(\beta N(\mathcal{H}_\nu - \mathcal{H}_\mu)).$$

To prove it, by Jensen's inequality we have

$$\log \int e^{\beta N(\mathcal{H}_\mu - \mathcal{H}_\nu)} \frac{e^{-\beta N \mathcal{H}_\mu}}{\int e^{-\beta N \mathcal{H}_\mu} d\lambda} d\lambda \geq \int \beta N(\mathcal{H}_\mu - \mathcal{H}_\nu) \frac{e^{-\beta N \mathcal{H}_\mu}}{\int e^{-\beta N \mathcal{H}_\mu} d\lambda} d\lambda.$$

We now bound

$$\mathbb{E}_\mu(N(\mathcal{H}_\nu - \mathcal{H}_\mu)) = \mathbb{E}_\mu(NX_\alpha^2) + \mathbb{E}_\mu(N\psi_{i,j}) + \mathbb{E}_\mu(N\psi^{(s)}) \quad (3.23)$$

in the following way. By Lemma 2.1, $\mathbb{E}_\mu(NX_\alpha^2) < c_\alpha(\log N)^2$; by Lemma 3.1 (together to (2.6)), $\mathbb{E}_\mu(N\psi_{i,j})$ is subexponentially small for any indices i and j ; finally $\mathbb{E}_\mu(N\psi^{(s)})$ is also subexponentially small by (3.3) (together to (2.6)). Altogether, we get that there is a constant $c > 0$ such that for any $N \geq 2$

$$\log Z_\nu \leq \log Z_\mu \leq \log Z_\nu + c(\log N)^2. \quad (3.24)$$

Let $(A_N)_{N \geq 1}$ be now a sequence of events exponentially small for μ . By (3.24) we have

$$\mathbb{P}_\nu(A_N) \leq e^{c(\log N)^2} \mathbb{P}_\mu(A_N),$$

so $(A_N)_{N \geq 1}$ is also exponentially small for ν .

Assume now that $(A_N)_{N \geq 1}$ is exponentially small for ν : there are constants $\delta, c_1, c_2 > 0$ such that for any N we have

$$\mathbb{P}_\nu(A_N) \leq c_1 e^{-c_2 N^\delta}.$$

Then for any t we have

$$\begin{aligned} \mathbb{P}_\mu(A_N) &= \mathbb{P}_\mu(A_N \cap \{\beta N(\mathcal{H}_\nu - \mathcal{H}_\mu) > t\}) + \mathbb{P}_\mu(A_N \cap \{\beta N(\mathcal{H}_\nu - \mathcal{H}_\mu) < t\}) \\ &\leq \mathbb{P}_\mu(\{\beta N(\mathcal{H}_\nu - \mathcal{H}_\mu) > t\}) + e^t \mathbb{P}_\nu(A_N), \end{aligned}$$

where we used $Z_\nu < Z_\mu$. Choosing $t = N^{\delta/2}$ makes the second term exponentially small, and the first one as well by using as previously Lemma 2.1, Lemma 3.1 and (3.3). \square

From the previously proved equivalence of the measures μ and ν , we can easily obtain rigidity of the particles at scale $N^{-1/2}$,

Proposition 3.7 *For any $\alpha, \varepsilon > 0$, there are constants $\delta, c_1, c_2 > 0$ such that for any $N \geq 1$ and $k \in \llbracket \alpha N, (1 - \alpha)N \rrbracket$,*

$$\mathbb{P}_\mu \left(|\lambda_k - \gamma_k| > N^{-\frac{1}{2} + \varepsilon} \right) \leq c_1 e^{-c_2 N^\delta}.$$

Proof. From Lemma 3.5 about the convexity of \mathcal{H}_ν , we get by the classical Bakry-Émery criterion [3] that ν satisfies a logarithmic Sobolev inequality with constant of order $1/N$, so by Herbst's lemma concentration at scale $N^{-1/2}$ holds for individual particles for ν : there is a constant $c > 0$ such that for any $N \geq 1$, $k \in \llbracket 1, N \rrbracket$ and $x > 0$,

$$\mathbb{P}_\nu \left(|\lambda_k - \mathbb{E}_\nu(\lambda_k)| > xN^{-\frac{1}{2}} \right) \leq e^{-cx}.$$

By Lemma 3.6, this implies that for some constants $\delta, c_1, c_2 > 0$,

$$\mathbb{P}_\mu \left(|\lambda_k - \mathbb{E}_\nu(\lambda_k)| > N^{-\frac{1}{2}+\varepsilon} \right) \leq c_1 e^{-c_2 N^\delta}. \quad (3.25)$$

From the above equation we get $|\mathbb{E}_\nu(\lambda_k) - \mathbb{E}_\mu(\lambda_k)|$ is of order at most $N^{-1/2+\varepsilon}$, so (3.25) holds when replacing $\mathbb{E}_\nu(\lambda_k)$ by $\mathbb{E}_\mu(\lambda_k)$, proving concentration at scale $N^{-1/2}$ for μ .

Define $\gamma_k^{(N)}$ by

$$\int_{-\infty}^{\gamma_k^{(N)}} \rho_1^{(N)} = \frac{k}{N}. \quad (3.26)$$

The proof will be complete if we can prove that for any $\varepsilon > 0$ and $k \in \llbracket \alpha N, (1-\alpha)N \rrbracket$, for large enough N we have

$$|\gamma_k^{(N)} - \gamma_k| < N^{-1/2+\varepsilon}, \quad (3.27)$$

By Lemma 2.2, $|m_N - m| \rightarrow 0$ for $\eta > N^{-1/2+\varepsilon}$, because on this domain $\frac{1}{N^2}k_N \rightarrow 0$, as concentration at scale $N^{-1/2}$ holds for μ . So using Lemma 2.3 we finally get that (3.27) holds, finishing the proof. \square

4 The multiscale analysis

The purpose of this paragraph is to prove the following proposition: if rigidity holds at scale N^{-1+a} , it holds also at scale $N^{-1+\frac{3}{4}a}$. The argument very closely follows Section 3.3 of [6] and we will just explain the modifications.

Proposition 4.1 *Assume that for some $a \in (0, 1)$ the following property holds: for any $\alpha, \varepsilon > 0$, there are constants $\delta, c_1, c_2 > 0$ such that for any $N \geq 1$ and $k \in \llbracket \alpha N, (1-\alpha)N \rrbracket$,*

$$\mathbb{P}_\mu \left(|\lambda_k - \gamma_k| > N^{-1+a+\varepsilon} \right) \leq c_1 e^{-c_2 N^\delta}. \quad (4.1)$$

Then the same property holds also replacing a by $3a/4$: for any $\alpha, \varepsilon > 0$, there are constants $\delta, c_1, c_2 > 0$ such that for any $N \geq 1$ and $k \in \llbracket \alpha N, (1-\alpha)N \rrbracket$, we have

$$\mathbb{P}_\mu \left(|\lambda_k - \gamma_k| > N^{-1+\frac{3}{4}a+\varepsilon} \right) \leq c_1 e^{-c_2 N^\delta}.$$

Proof of Theorem 1.1. This is an immediate consequence of the initial estimate, Proposition 3.7, and iterations of Proposition 4.1. \square

As in Section 3.3 of [6], two steps are required in the proof of the above Proposition 4.1. First we will prove that concentration holds at the smaller scale $N^{-1+\frac{a}{2}}$.

Proposition 4.2 *Assume that (4.1) holds. Then for any $\alpha > 0$ and $\varepsilon > 0$, there are constants $c_1, c_2, \delta > 0$ such that for any $N \geq 1$ and $k \in \llbracket \alpha N, (1 - \alpha)N \rrbracket$,*

$$\mathbb{P}_\mu \left(|\lambda_k - \mathbb{E}_\mu(\lambda_k)| > \frac{N^{\frac{\alpha}{2} + \varepsilon}}{N} \right) \leq c_1 e^{-c_2 N^\delta}. \quad (4.2)$$

After the better concentration from this proposition, the rigidity can be improved to the scale $N^{-1 + \frac{3\alpha}{4}}$.

Proposition 4.3 *Assume that (4.1) holds. Then for any $\alpha > 0$ and $\varepsilon > 0$, there is a constant $c > 0$ such that for any $N \geq 1$ and $k \in \llbracket \alpha N, (1 - \alpha)N \rrbracket$,*

$$\left| \gamma_k^{(N)} - \gamma_k \right| \leq c \frac{N^{\frac{3\alpha}{4} + \varepsilon}}{N}$$

where $\gamma_k^{(N)}$ is defined in (3.26).

Propositions 4.2 and 4.3 are the equivalent versions of Propositions 3.12 and 3.13 of [6] with no convexity assumption on V . Proposition 4.1 can be proved exactly in the same way as Proposition 3.11 [6] by using Propositions 4.2 and 4.3. Notice that this argument does not use the convexity of V . We now explain the proof of Propositions 4.2 and 4.3.

The convexity of V is used critically in the proof of Proposition 3.12 of [6]. Our measure μ is not convex, but thanks to Lemma 3.6, it has the same exponentially small events as the measure $\nu^{(s, \ell)}$ for any fixed $s, \ell > 0$. Hence it suffices to prove (4.2) with μ replaced by $\nu^{(s, \ell)}$. Choose an appropriate s, ℓ such that the Hamiltonian of $\nu = \nu^{(s, \ell)}$ is convex (Lemma 3.5). Then the proof of (4.2) with μ replaced by ν is identical to the proof of Proposition 3.12 of [6] since the measure ν is now convex.

For the proof of Proposition 4.3, we can follow the proof of Proposition 3.13 in [6] line by line. At a single place, in estimating the second term on the r.h.s. of (3.51) in [6], the spectral gap inequality for μ (Eq. (3.12) in [6]) was used, but the necessary estimate immediately follows from Proposition 3.7.

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References

- [1] S. Alberverio, L. Pastur, and M. Shcherbina, *On the $1/n$ expansion for some unitary invariant ensembles of random matrices*, Commun. Math. Phys. **224** (2001), 271–305.
- [2] G. W. Anderson, A. Guionnet, and O. Zeitouni, *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics, vol. 118, Cambridge University Press, Cambridge, 2010.
- [3] D. Bakry and M. Émery, *Diffusions hypercontractives*, Séminaire de probabilités XIX **1123** (1983), no. 84, 117–206.
- [4] G. Ben Arous and A. Guionnet, *Large deviations for Wigner’s law and Voiculescu’s non-commutative entropy*, Probab. Theory Related Fields **108** (1997), no. 4, 517–542.
- [5] P. Bleher and A. Its, *Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model*, Ann. of Math. (2) **150** (1999), no. 1, 185–266.
- [6] P. Bourgade, L. Erdős, and H.-T. Yau, *Universality of general β -ensembles*, Preprint [arXiv:0907.5605](https://arxiv.org/abs/0907.5605) (2011).

- [7] A. Boutet de Monvel, L. Pastur, and M. Shcherbina, *On the statistical mechanics approach in the Random Matrix Theory. Integrated Density of States*, J. Stat. Phys. **79** (1995), 585–611.
- [8] H. Brascamp and E. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler Theorems, Including Inequalities for Log Concave Functions, and with an Application to the Diffusion Equation*, Journal of Functional Analysis **22** (1976), 366–389.
- [9] P. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, Courant Lecture Notes in Mathematics, vol. 3, American Mathematical Society, Providence, RI, Cambridge, 1999.
- [10] P. Deift and D. Gioev, *Random matrix theory: invariant ensembles and universality*, Courant Lecture Notes in Mathematics, vol. 18, Courant Institute of Mathematical Sciences, New York, 2009.
- [11] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, *Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*, Comm. Pure Appl. Math. **52** (1999), no. 11, 1335–1425.
- [12] ———, *Strong asymptotics of orthogonal polynomials with respect to exponential weights*, Comm. Pure Appl. Math. **52** (1999), no. 12, 1491–1552.
- [13] I. Dumitriu and A. Edelman, *Matrix models for beta ensembles*, J. Math. Phys. **43** (2002), 5830–5847.
- [14] L. Erdős, A. Knowles, H.-T. Yau, and J. Yin, *Spectral Statistics of Erdős-Rényi Graphs II: Eigenvalue Spacing and the Extreme Eigenvalues*, Preprint [arXiv:1103.3869](https://arxiv.org/abs/1103.3869) (2011).
- [15] L. Erdős, J.-A. Ramírez, B. Schlein, and H.-T. Yau, *Universality of sine-kernel for Wigner matrices with a small Gaussian perturbation*, Electronic Journal of Probability **15** (2010), no. 18, 526–603.
- [16] B. Eynard, *Master loop equations, free energy and correlations for the chain of matrices*, J. High Energy Phys. **11** (2003).
- [17] K. Johansson, *On fluctuations of eigenvalues of random Hermitian matrices*, Duke Math. J. **91** (1998), no. 1, 151–204.
- [18] T. Kriecherbauer and M. Shcherbina, *Fluctuations of eigenvalues of matrix models and their applications*, Preprint [arXiv:1003.6121](https://arxiv.org/abs/1003.6121) (2011).
- [19] A. B. J. Kuijlaars and K. T.-R. McLaughlin, *Generic behavior of the density of states in random matrix theory and equilibrium problems in the presence of real analytic external fields*, Comm. Pure Appl. Math. **53** (2000), no. 6, 736–785.
- [20] D. Lubinsky, *A new approach to universality limits involving orthogonal polynomials*, Ann. of Math. (2) **170** (2009), no. 2, 915–939.
- [21] M. L. Mehta, *Random matrices*, 2nd ed., Academic Press Inc., Boston, MA, 1991.
- [22] L. Pastur and M. Shcherbina, *Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles*, J. Statist. Phys. **86** (1997), no. 1-2, 109–147.
- [23] ———, *Bulk universality and related properties of Hermitian matrix models*, J. Stat. Phys. **130** (2008), no. 2, 205–250.
- [24] ———, *Eigenvalue Distribution of Large Random Matrices*, Amer. Math. Society, Providence, RI, 2011.
- [25] M. Shcherbina, *Orthogonal and Symplectic Matrix Models: Universality and Other Properties*, Comm. Math. Phys. **307** (2011), no. 3, 761–790.
- [26] B. Valkó and B. Virág, *Continuum limits of random matrices and the Brownian carousel*, Invent. Math. **177** (2009), no. 3, 463–508.