Sparse Recovery Beyond Compressed Sensing

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Acknowledgements

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Separable Nonlinear Inverse Problems

Sparse Recovery for SNL Problems

Super-resolution

Deconvolution

SNL Problems with Correlation Decay

Separable nonlinear inverse (SNL) problems

Aim: estimate parameters $\theta_1, \ldots, \theta_s \in \mathbb{R}^d$ from data $y \in \mathbb{R}^n$

Relation between data and each θ_j governed by nonlinear function ϕ

Contributions of $\theta_1, \ldots, \theta_s$ combine linearly with unknown coeffs $c \in \mathbb{R}^s$

$$
\begin{aligned}
y &= \sum_{j=1}^{s} c\,(j)\,\phi(\theta_j) \\
&= \begin{bmatrix} \phi(\theta_1) & \phi(\theta_2) & \cdots & \phi(\theta_s) \end{bmatrix} c
\end{aligned}
$$

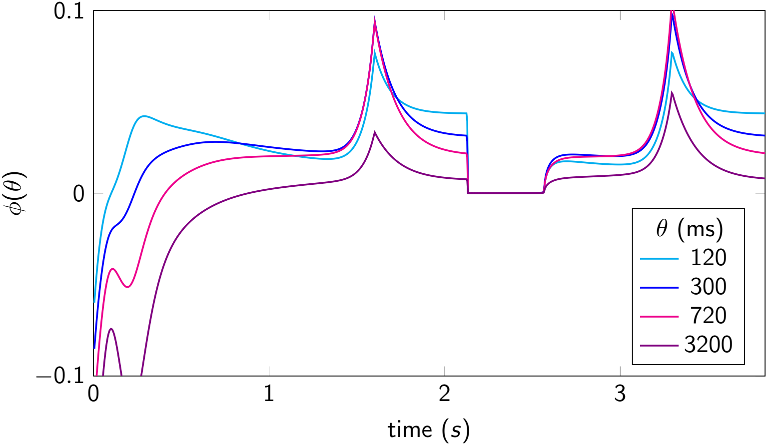
$n > s$, easy if we know $\theta_1, \ldots, \theta_s$

SNL problems

- Super-resolution
- Deconvolution
- Source localization in EEG
- Direction of arrival in radar / sonar
- Magnetic-resonance fingerprinting

Magnetic-resonance fingerprinting (Ma et al, 2013)

Goal: Estimate magnetic relaxation-time constants of tissues in a voxel



Separable Nonlinear Inverse Problems

Sparse Recovery for SNL Problems

Super-resolution

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SNL Problems with Correlation Decay

Methods to tackle SNL problems

- Nonlinear least-squares solved by descent methods
  Drawback: local minima

- Prony-based / Finite-rate of innovation
  Drawback: challenging to apply beyond super-resolution

- Reformulate as sparse-recovery problem
  (drawbacks discussed at the end)

Linearization

Linearize problem by lifting to a higher-dimensional space

True parameters: $\theta_{T_1}, \ldots, \theta_{T_s}$

Grid of parameters: $\theta_1, \ldots, \theta_N$, $N >> n$

$$y = \begin{bmatrix} \phi(\theta_1) & \cdots & \phi(\theta_{T_1}) & \cdots & \phi(\theta_{T_s}) & \cdots & \phi(\theta_N) \end{bmatrix} \begin{bmatrix} 0 \\ \ldots \\ c\,(1) \\ \ldots \\ c\,(s) \\ 0 \end{bmatrix}$$

$$= \sum_{j=1}^{s} c\,(j)\,\phi(\theta_{T_j})$$

Sparse Recovery for SNL Problems

Find a sparse $\tilde{c}$ such that

$$y = \Phi_{\text{grid}} \tilde{c}$$

Underdetermined linear inverse problem with sparsity prior

Popular approach: ℓ_1-norm minimization

$$
\begin{aligned}
\text{minimize} \quad & ||\tilde{c}||_1 \\
\text{subject to} \quad & \Phi_{\text{grid}}\tilde{c} = y
\end{aligned}
$$

Popular approach: ℓ_1-norm minimization

- Deconvolution:
  Deconvolution with the ℓ_1 norm, Taylor et al (1979)

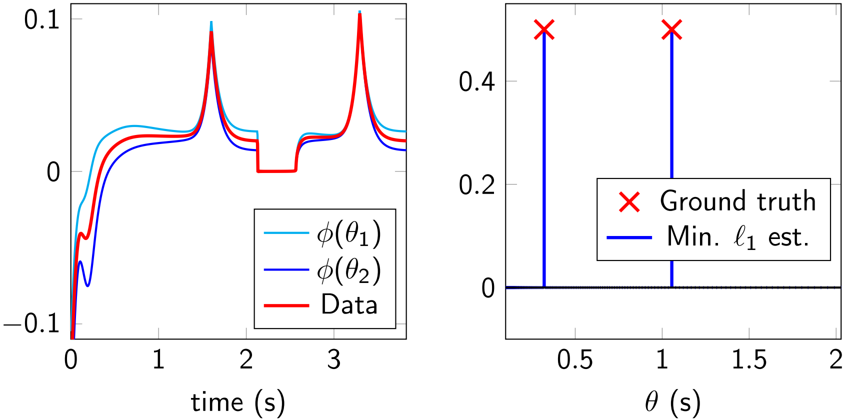
- EEG:
  Selective minimum-norm solution of the biomagnetic inverse problem, Matsuura and Okabe (1995)

- Direction-of-arrival in radar / sonar:
  A sparse signal reconstruction perspective for source localization with sensor arrays, Malioutov et al (2005)

- and many, many others...

Magnetic-resonance fingerprinting

Multicompartment magnetic resonance fingerprinting
(Tang, F., Lannuzel, Bernstein, Lattanzi, Cloos, Knoll, Asslaender 2018)



Main question

Under what conditions can SNL problems be solved by ℓ_1-norm minimization?

Continuous dictionary

Analysis should apply to arbitrarily fine grids

We model the coefficients / parameter values as an atomic measure

$$x := \sum_{j=1}^{s} c\,(j)\,\delta_{\theta_{T_j}}$$

$$y = \sum_{j=1}^{s} c\,(j)\,\phi(\theta_{T_j})$$

$$= \int \phi(\theta)\,x(\,\mathrm{d}\theta) = \Phi x$$

Intuitively, Φ is a continuous dictionary with n rows

Sparse Recovery for SNL Problems

Find a sparse $\tilde{x}$ such that

$$y = \int \phi(\theta)\, \tilde{x}(\,\mathrm{d}\theta)$$

(Extremely) underdetermined linear inverse problem with sparsity prior

Total-variation norm

Continuous counterpart of the ℓ_1 norm

Not the total variation of a piecewise-constant function

$$||c||_1 = \sup_{||\vec{v}||_\infty \leq 1} \langle v, c \rangle$$

$$||x||_{\mathrm{TV}} = \sup_{f \in \mathbb{C}^{[0,1]},\ ||f||_\infty \leq 1} \int_{[0,1]} f(t)\, x(\mathrm{d}t)$$

If $x = \sum_j c_j \delta_{\theta_j}$ then $||x||_{\mathrm{TV}} = ||c||_1$

Main question

For an SNL problem, when does

$$\begin{aligned} &\text{minimize} && \|\tilde{x}\|_{\mathsf{TV}} \\ &\text{subject to} && \int \phi(\theta)\,\tilde{x}(\,\mathrm{d}\theta) = y \end{aligned}$$

achieve exact recovery?

Wait, isn't this just compressed sensing?

Compressed sensing

Recover s-sparse vector x of dimension m from $n < m$ measurements

$$y = Ax$$

Key assumption: A is random, and hence satisfies restricted-isometry properties with high probability

Restricted isometry property (Candès, Tao 2006)

An $m \times n$ matrix A satisfies the restricted isometry property (RIP) if there exists $0 < \kappa < 1$ such that for any s-sparse vector $\mathbf{x}$

$$(1 - \kappa) \left|\left|x\right|\right|_2 \leq \left|\left|Ax\right|\right|_2 \leq (1 + \kappa) \left|\left|x\right|\right|_2$$

$2s$-RIP implies that for any s-sparse signals x_1, x_2

$$\left|\left|Ax_2 - Ax_1\right|\right|_2$$

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$2s$-RIP implies that for any s-sparse signals x_1, x_2

$$\begin{aligned} ||Ax_2 - Ax_1||_2 &= ||A\,(x_2 - x_1)||_2 \\ &\geq (1-\kappa)\,||x_2 - x_1||_2 \end{aligned}$$

Separable nonlinear problems

If ϕ is smooth, nearby columns in

$$\Phi_{\text{grid}} := \begin{bmatrix} \phi(\theta_1) & \phi(\theta_2) & \cdots & \phi(\theta_N) \end{bmatrix}$$

are highly correlated so RIP does not hold!

There are x_1, x_2 such that $Ax_1 \approx Ax_2$

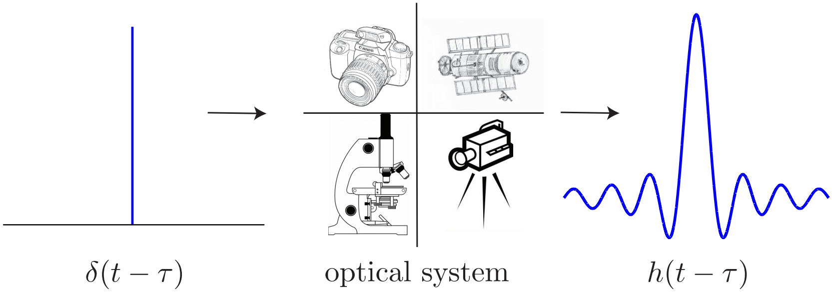
Sparsity is not enough, we need additional restrictions!

Super-resolution

Joint work with Emmanuel Candès (Stanford)

Limits of resolution in imaging

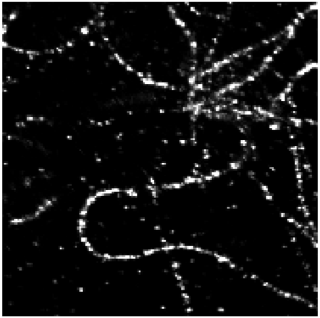
The resolving power of lenses, however perfect, is limited (**Lord Rayleigh**)



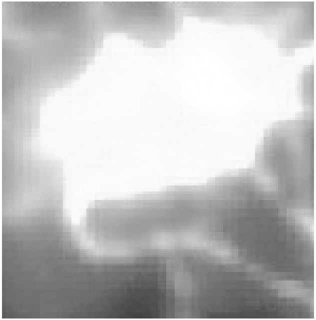
Diffraction imposes a fundamental limit on the resolution of optical systems

Fluorescence microscopy

Data



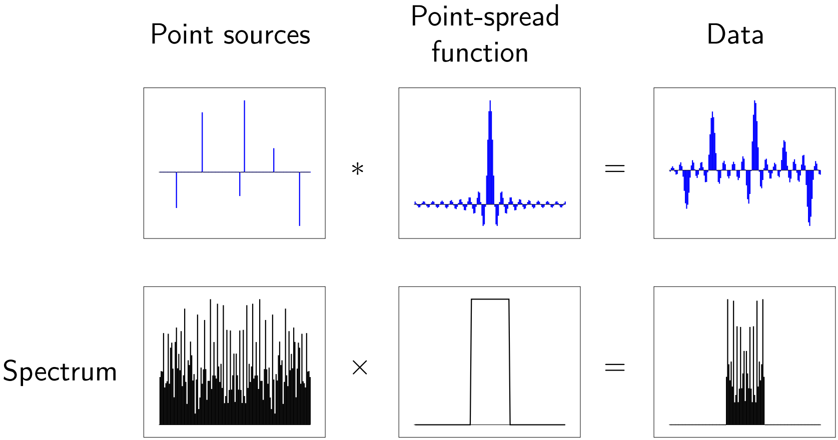
Point sources



Low-pass blur

(Figures courtesy of V. Morgenshtern)

Sensing model for super-resolution



Super-resolution

s sources with locations $\theta_1, \ldots, \theta_s$, modeled as superposition of spikes

$$x = \sum_j c(j) \delta_{\theta_j} \qquad c_j \in \mathbb{C}, \, \theta_j \in T \subset [0, 1]$$

We observe Fourier coefficients up to cut-off frequency f_c

$$\begin{aligned} y(k) &= \int_0^1 \exp(-i2\pi kt) \, x(\mathrm{d}t) \\ &= \sum_{j=1}^{s} c(j) \exp(-i2\pi k\theta_j) \end{aligned}$$

SNL problem where

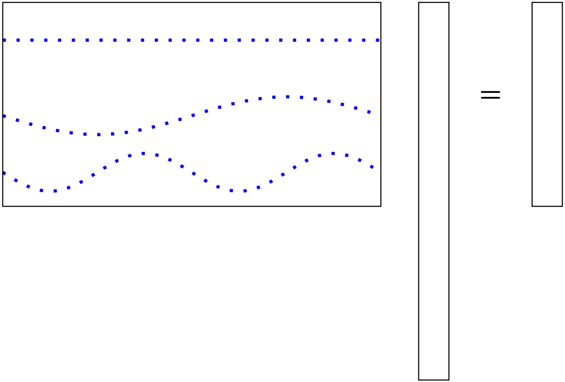
$$\phi(\theta_j) = \begin{bmatrix} \exp(-i2\pi\theta_j(-f_c)) \\ \cdots \\ \exp(-i2\pi\theta_j f_c) \end{bmatrix}$$

Fundamental questions

1. Is the problem well posed?

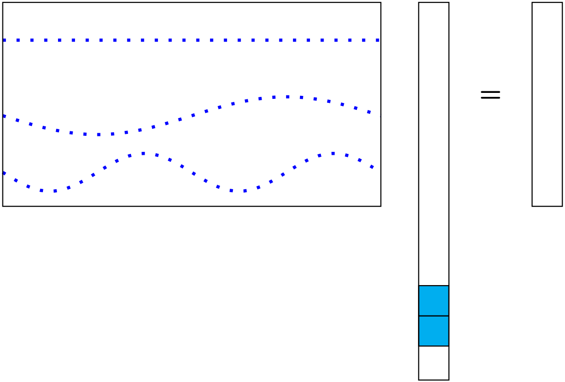
2. Does TV-norm minimization work?

Is the problem well posed?



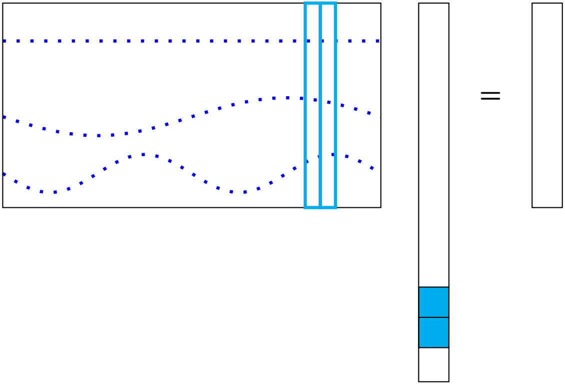
Measurement operator = low-pass samples with cut-off frequency f_c

Is the problem well posed?



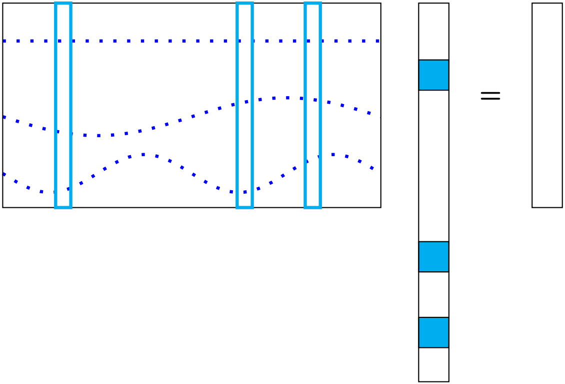
Effect of measurement operator on sparse vectors?

Is the problem well posed?



Submatrix can be very ill conditioned!

Is the problem well posed?

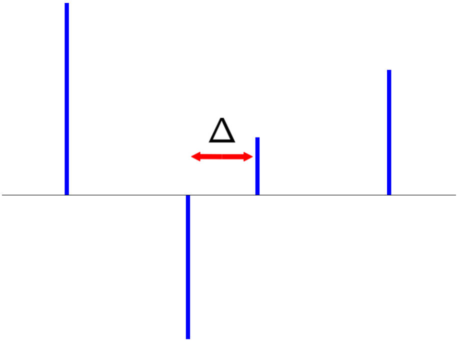


If support is spread out there is hope

Minimum separation

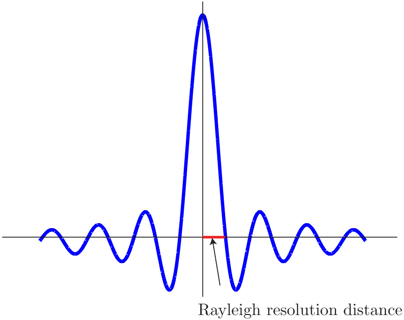
The minimum separation Δ of the support of x is

$$\Delta = \inf_{(\theta,\theta') \in \text{support}(x)\,:\,\theta \neq \theta'} |\theta - \theta'|$$



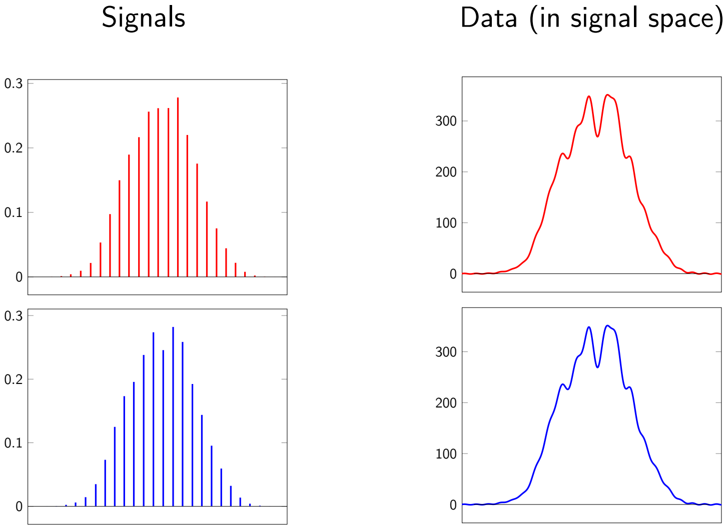
Conditioning of submatrix with respect to Δ

- If $\Delta < 1/f_c$ the problem is ill posed
- If $\Delta > 1/f_c$ the problem becomes well posed
- Proved asymptotically by Slepian and non-asymptotically by Moitra

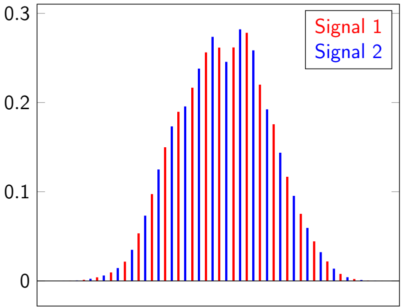


$1/f_c$ is the diameter of the main lobe of the point-spread function (twice the Rayleigh distance)

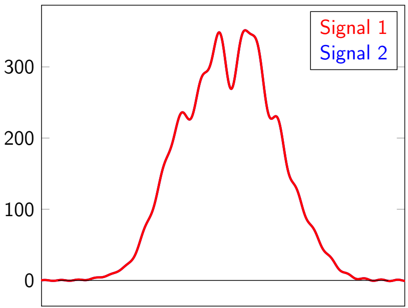
Example: 25 spikes, $f_c = 10^3$, $\Delta = 0.8/f_c$



Example: 25 spikes, $f_c = 10^3$, $\Delta = 0.8/f_c$



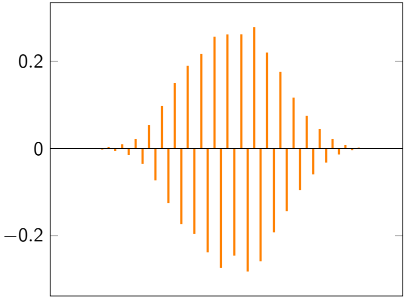
Signals



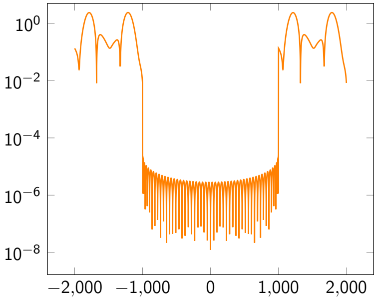
Data (in signal space)

Example: 25 spikes, $f_c = 10^3$, $\Delta = 0.8/f_c$

The difference is almost in the null space of the measurement operator



Difference



Spectrum

Theoretical questions

1. Is the problem well posed?

2. Does TV-norm minimization work?

Super-resolution via TV-norm minimization

$$\begin{aligned} &\text{minimize} && ||\tilde{x}||_{\mathsf{TV}} \\ &\text{subject to} && \int \phi(\theta)\,\tilde{x}(\,\mathrm{d}\theta) = y \end{aligned}$$

Dual certificate for TV-norm minimization

$v \in \mathbb{R}^n$ is a dual certificate associated to

$$x = \sum_j c_j \delta_{\theta_j} \qquad c_j \in \mathbb{R}, \; \theta_j \in T$$

if

$$Q(\theta) := v^T \phi(\theta)$$

$$Q(\theta_j) = \text{sign}(c_j) \qquad \text{if } \theta_j \in T$$

$$|Q(\theta)| < 1 \qquad \text{if } \theta \notin T$$

Dual variable guaranteeing that $||x||_{\text{TV}}$ is optimal

Dual certificate

For any $x + h$ such that $\int \phi(\theta)\, h(\,\mathrm{d}\theta) = 0$

$$||x + h||_{\mathrm{TV}} = \sup_{||f||_\infty \leq 1} \int_{[0,1]} f(\theta)\, x(\,\mathrm{d}\theta) + \int_{[0,1]} f(\theta)\, h(\,\mathrm{d}\theta)$$

Dual certificate

For any $x + h$ such that $\int \phi(\theta)\, h(\,\mathrm{d}\theta) = 0$

$$\begin{aligned} ||x + h||_{\mathrm{TV}} &= \sup_{||f||_\infty \leq 1} \int_{[0,1]} f\,(\theta)\, x\,(\,\mathrm{d}\theta) + \int_{[0,1]} f\,(\theta)\, h\,(\,\mathrm{d}\theta) \\ &\geq \int_{[0,1]} Q\,(\theta)\, x\,(\,\mathrm{d}\theta) + \int_{[0,1]} Q\,(\theta)\, h\,(\,\mathrm{d}\theta) \end{aligned}$$

Dual certificate

For any $x+h$ such that $\int \phi(\theta)\, h(\,\mathrm{d}\theta) = 0$

$$
\begin{aligned}
\|x+h\|_{\mathrm{TV}} &= \sup_{\|f\|_\infty \le 1} \int_{[0,1]} f\,(\theta)\, x\,(\,\mathrm{d}\theta) + \int_{[0,1]} f\,(\theta)\, h\,(\,\mathrm{d}\theta) \\
&\geq \int_{[0,1]} Q\,(\theta)\, x\,(\,\mathrm{d}\theta) + \int_{[0,1]} Q\,(\theta)\, h\,(\,\mathrm{d}\theta) \\
&\geq \sum_{\theta_j \in T} \int_{[0,1]} Q\,(\theta_j)\, c_j \delta_{\theta_j}\,(\,\mathrm{d}\theta) + v^T \int_{[0,1]} \phi\,(\theta)\, h\,(\,\mathrm{d}\theta)
\end{aligned}
$$

Dual certificate

For any $x + h$ such that $\int \phi(\theta)\, h(\,\mathrm{d}\theta) = 0$

$$\begin{aligned}
||x + h||_{\mathrm{TV}} &= \sup_{||f||_\infty \leq 1} \int_{[0,1]} f(\theta)\, x(\,\mathrm{d}\theta) + \int_{[0,1]} f(\theta)\, h(\,\mathrm{d}\theta) \\
&\geq \int_{[0,1]} Q(\theta)\, x(\,\mathrm{d}\theta) + \int_{[0,1]} Q(\theta)\, h(\,\mathrm{d}\theta) \\
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&= ||x||_{\mathrm{TV}}
\end{aligned}$$

Dual certificate

For any $x + h$ such that $\int \phi(\theta)\, h(\,\mathrm{d}\theta) = 0$

$$\begin{aligned}
||x + h||_{\mathrm{TV}} &= \sup_{||f||_\infty \leq 1} \int_{[0,1]} f(\theta)\, x(\,\mathrm{d}\theta) + \int_{[0,1]} f(\theta)\, h(\,\mathrm{d}\theta) \\
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&\geq \sum_{\theta_j \in T} \int_{[0,1]} Q(\theta_j)\, c_j \delta_{\theta_j}(\,\mathrm{d}\theta) + v^T \int_{[0,1]} \phi(\theta)\, h(\,\mathrm{d}\theta) \\
&= ||x||_{\mathrm{TV}}
\end{aligned}$$

Existence of Q for any sign pattern implies that x is the unique solution

Dual certificate for super-resolution

$v \in \mathbb{C}^n$ is a dual certificate associated to

$$x = \sum_j c_j \delta_{\theta_j} \qquad c_j \in \mathbb{C},\ \theta_j \in T$$

if

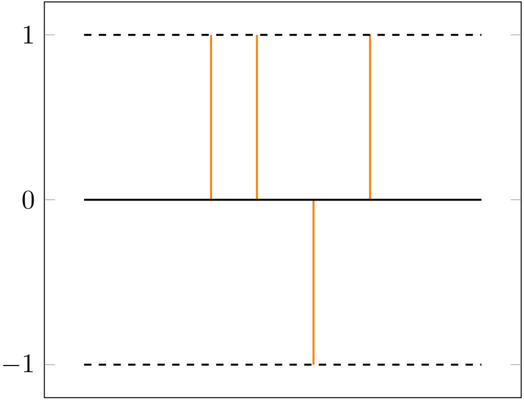
$$Q(\theta) := v^* \phi(\theta) = \sum_{k=-f_c}^{f_c} \overline{v_k} \exp(i 2\pi k \theta)$$

$$Q(\theta_j) = \operatorname{sign}(c_j) \qquad \text{if } \theta_j \in T$$

$$|Q(\theta)| < 1 \qquad \text{if } \theta \notin T$$

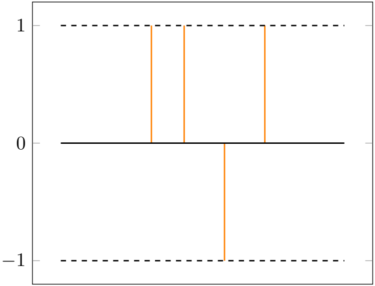
Linear combination of low pass sinusoids

Certificate for super-resolution



Aim: Interpolate sign pattern

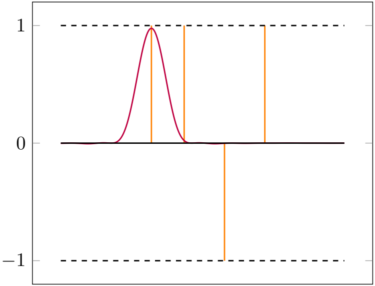
Certificate for super-resolution



Interpolation with a low-frequency fast-decaying kernel F

$$q(t) = \sum_{\theta_j \in T} \alpha_j \, F\left(t - \theta_j\right)$$

Certificate for super-resolution



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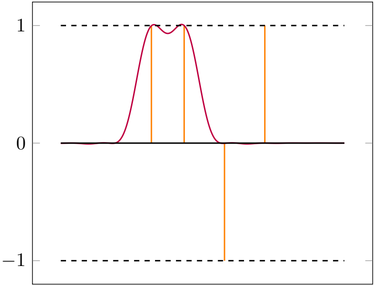


Interpolation with a low-frequency fast-decaying kernel F

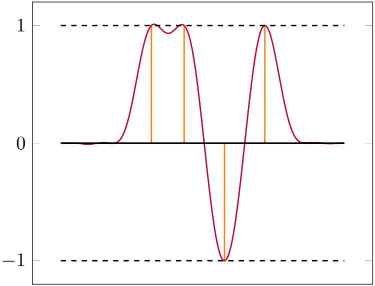
$$q(t) = \sum_{\theta_j \in T} \alpha_j \, F\left(t - \theta_j\right)$$

Certificate for super-resolution

Interpolation with a low-frequency fast-decaying kernel F

$$q(t) = \sum_{\theta_j \in T} \alpha_j \, F\left(t - \theta_j\right)$$

Certificate for super-resolution



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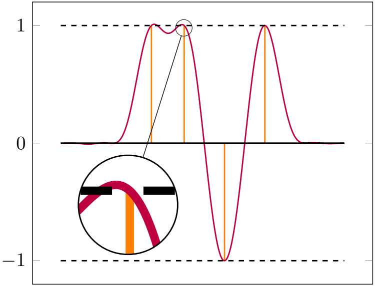
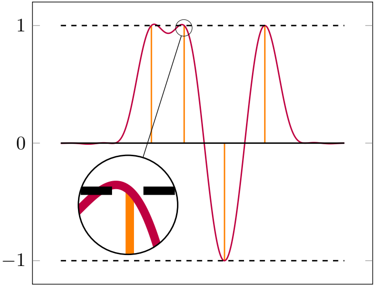


Technical detail: Magnitude of certificate locally exceeds 1

Certificate for super-resolution

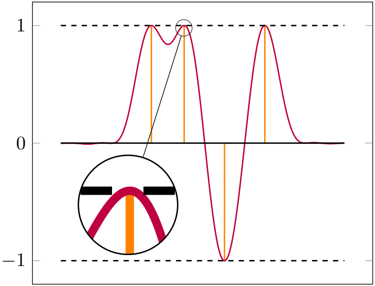


Technical detail: Magnitude of certificate locally exceeds 1

Solution: Add correction term and force derivative to vanish on support

$$Q(\theta) = \sum_{\theta_j \in T} \alpha_j \, F(\theta - \theta_j) \; + \; \beta_j \, F'(\theta - \theta_j)$$

Certificate for super-resolution



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Guarantees for super-resolution

Theorem [Candès, F. 2012]

If the minimum separation of the signal support obeys

$$\Delta \geq 2 \,/ f_c$$

then recovery via convex programming is exact

Theorem [Candès, F. 2012]

In 2D convex programming super-resolves point sources with a minimum separation of

$$\Delta \geq 2.38 \,/ f_c$$

where f_c is the cut-off frequency of the low-pass kernel

Guarantees for super-resolution

Theorem [F. 2016]

If the minimum separation of the signal support obeys

$$\Delta \geq 1.26 \,/f_c,$$

then recovery via convex programming is exact

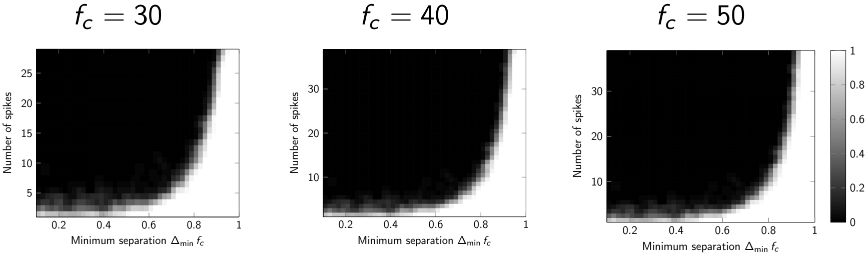
Theorem [Candès, F. 2012]

In 2D convex programming super-resolves point sources with a minimum separation of

$$\Delta \geq 2.38 \,/f_c$$

where f_c is the cut-off frequency of the low-pass kernel

Numerical evaluation of minimum separation



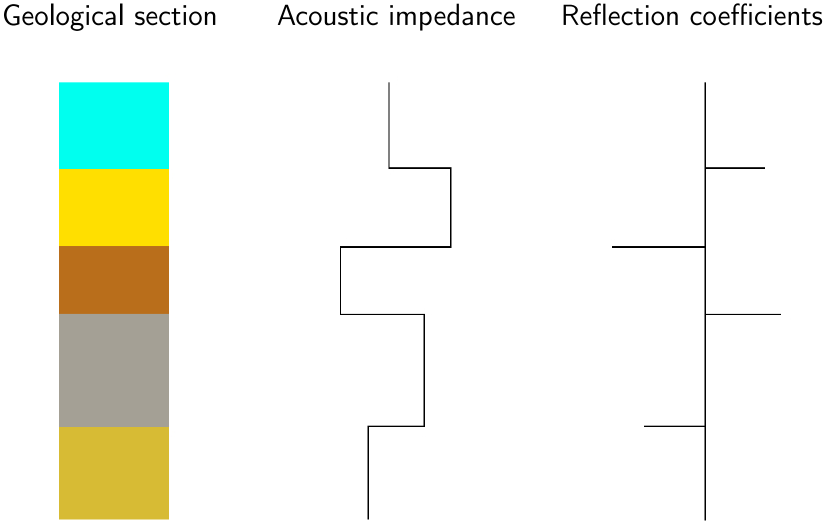
Numerically TV-norm minimization succeeds if $\Delta \geq \frac{1}{f_c}$

Deconvolution

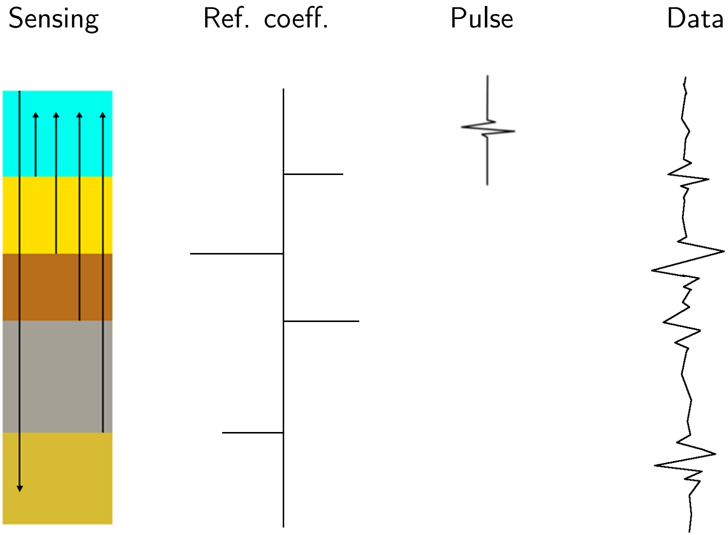
Joint work with Brett Bernstein (Courant)

Seismology

Reflection seismology

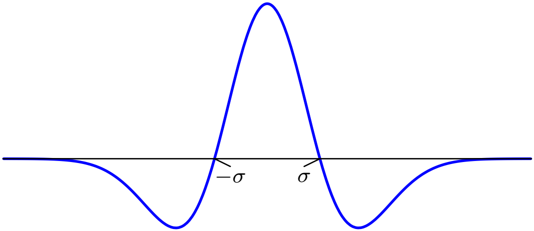


Reflection seismology

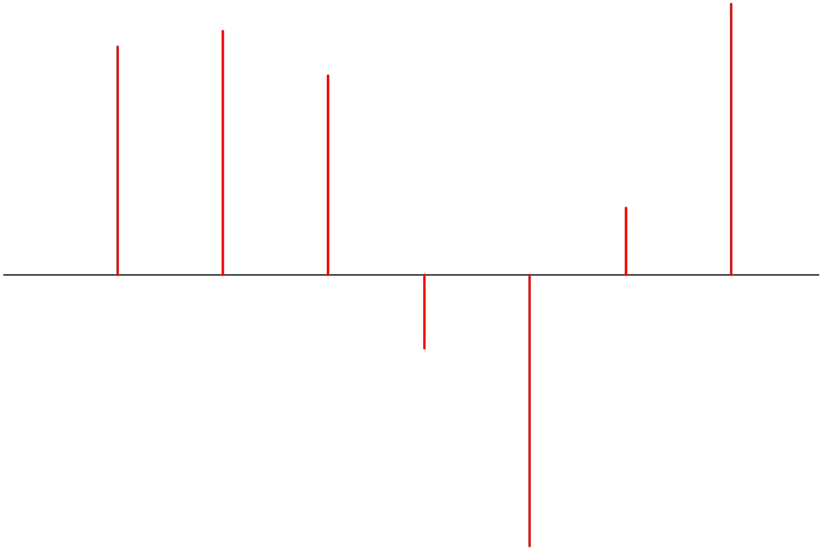


Data ≈ convolution of pulse and reflection coefficients

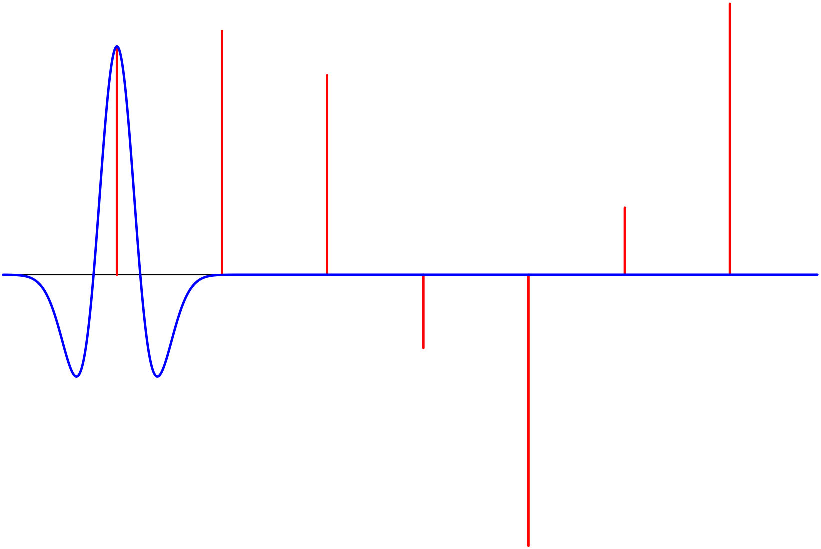
Model for the pulse: Ricker wavelet



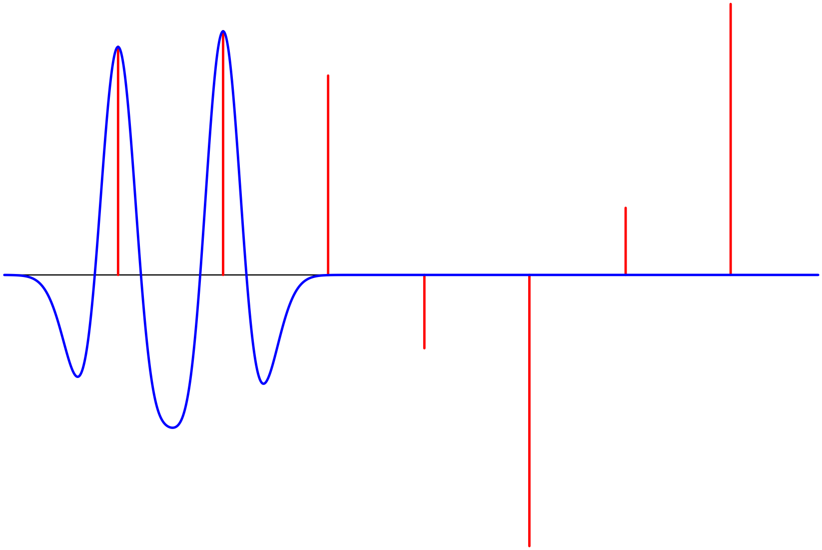
Toy model for reflection seismology



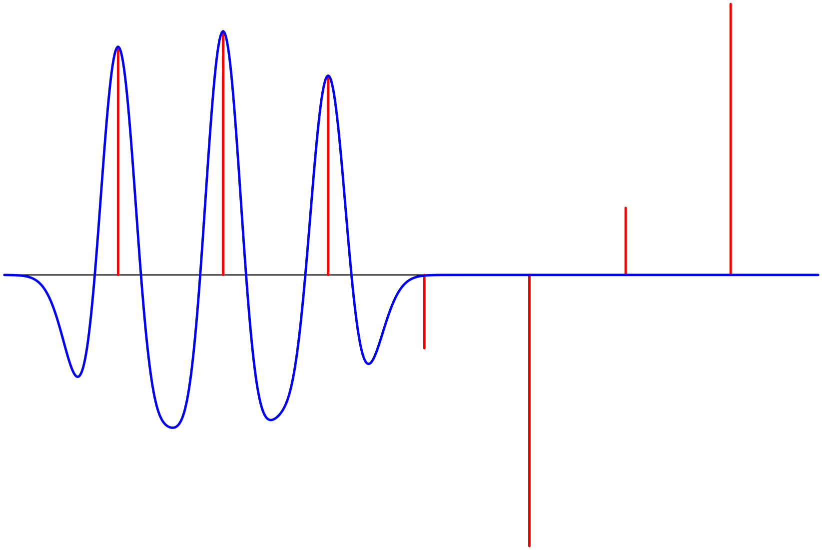
Toy model for reflection seismology



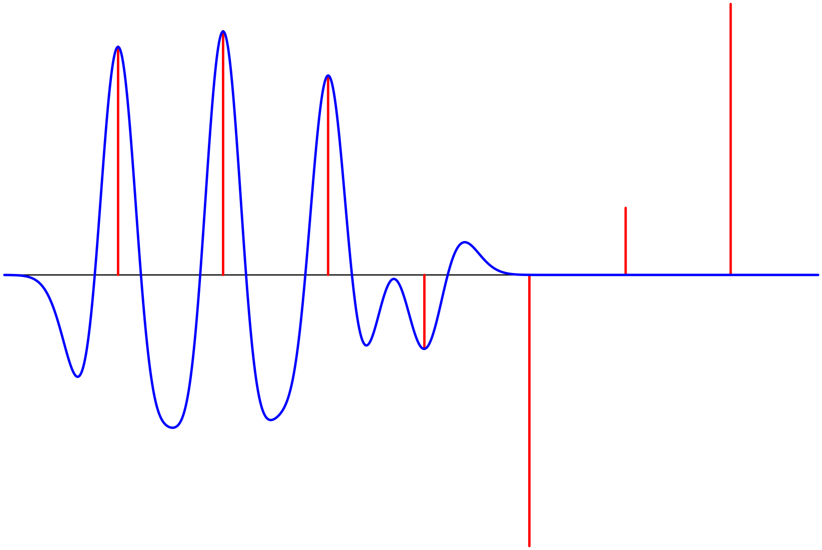
Toy model for reflection seismology



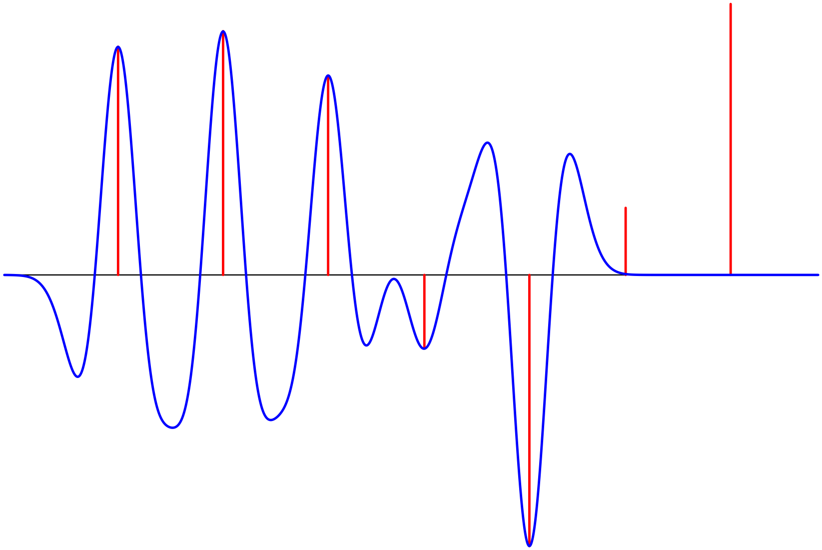
Toy model for reflection seismology



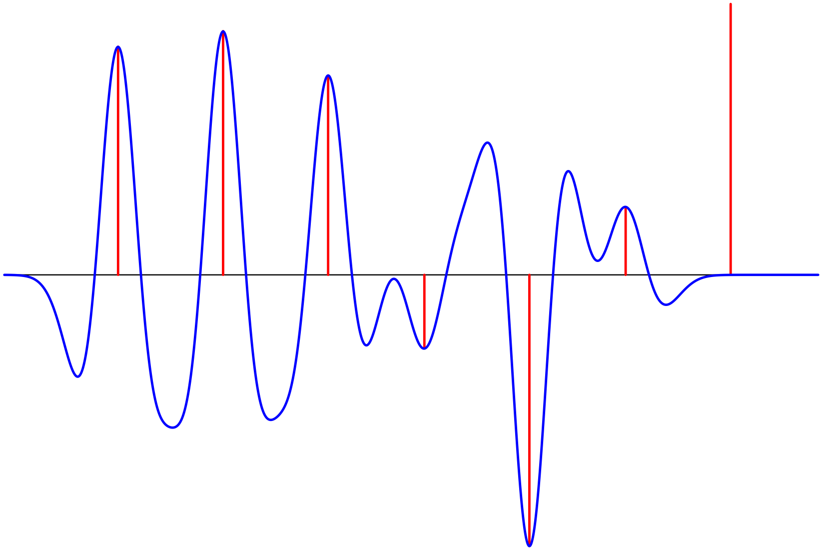
Toy model for reflection seismology



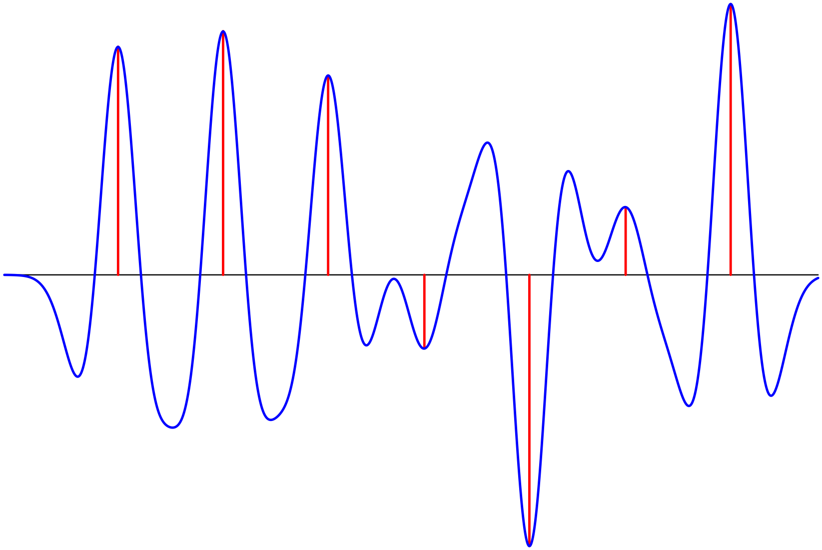
Toy model for reflection seismology



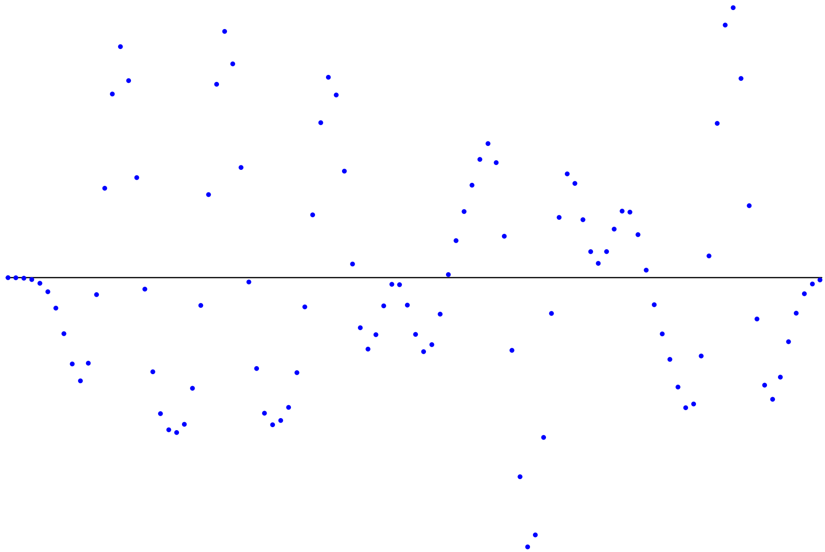
Toy model for reflection seismology



Toy model for reflection seismology



Toy model for reflection seismology



Deconvolution

s sources with locations $\theta_1, \ldots, \theta_s$, modeled as superposition of spikes

$$x = \sum_j c(j)\delta_{\theta_j} \qquad c_j \in \mathbb{R},\ \theta_j \in T \subset [0,1]$$

We observe samples of convolution with kernel K

$$\begin{aligned} y(k) &= (K * x)(s_k) \\ &= \sum_{j=1}^{s} c(j)\, K(s_k - \theta_j) \end{aligned}$$

SNL problem where

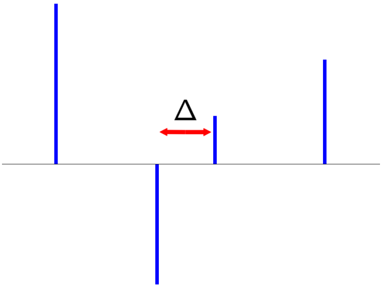
$$\phi(\theta_j) = \begin{bmatrix} K(s_1 - \theta_j) \\ \cdots \\ K(s_n - \theta_j) \end{bmatrix}$$

Theoretical questions

1. Is the problem well posed?

2. Does TV-norm minimization work?

Minimum separation



Kernels are approximately low-pass

The support cannot be too clustered

Sampling proximity

We need two samples per spike

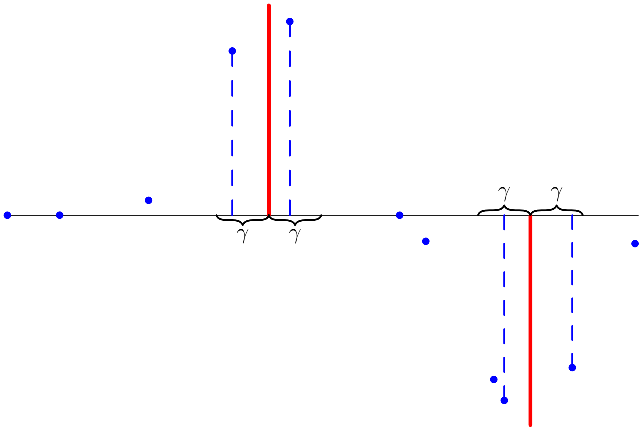
Convolution kernel decays: at least two samples close to each spike

Samples S and support T have sample proximity γ if for every $\theta_i \in T$ there exist $s_i, s_i' \in S$ such that

$$|\theta_i - s_i| \leq \gamma \quad \text{and} \quad \left|\theta_i - s_i'\right| \leq \gamma$$

We consider arbitrary non-uniform sampling patterns with fixed γ

Sampling proximity



Theoretical questions

1. Is the problem well posed?
2. Does TV-norm minimization work?

Deconvolution via TV-norm minimization

$$\begin{aligned} &\text{minimize} && ||\tilde{x}||_{\mathsf{TV}} \\ &\text{subject to} && \int \phi(\theta)\,\tilde{x}(\,\mathrm{d}\theta) = y \end{aligned}$$

Dual certificate for SNL problems

$v \in \mathbb{R}^n$ is a dual certificate associated to

$$x = \sum_j c_j \delta_{\theta_j} \qquad c_j \in \mathbb{R},\ \theta_j \in T$$

if

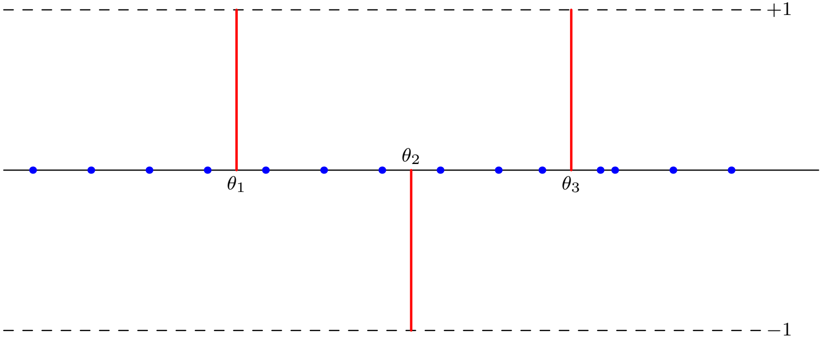
$$Q(\theta) := v^T \phi(\theta) = \sum_{k=1}^{n} v_k K(s_k - \theta)$$

$$Q(\theta_j) = \operatorname{sign}(c_j) \qquad \text{if } \theta_j \in T$$

$$|Q(\theta)| < 1 \qquad \text{if } \theta \notin T$$

Linear combination of shifted copies of K fixed at the samples

Certificate for deconvolution



Certificate construction

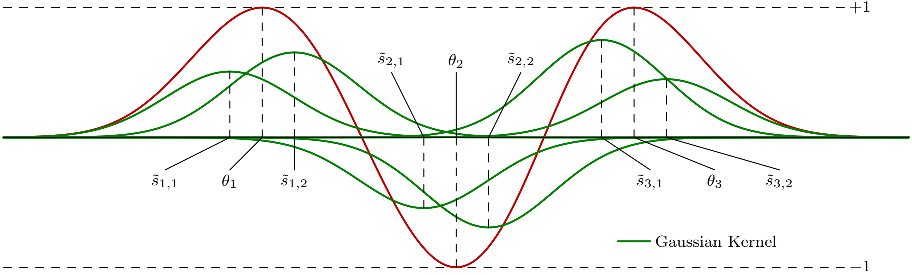
Only use subset $\widetilde{S}$ containing 2 samples close to each spike

$$Q(\theta) = \sum_{s_j \in \widetilde{S}} v_j K(s_j - \theta)$$

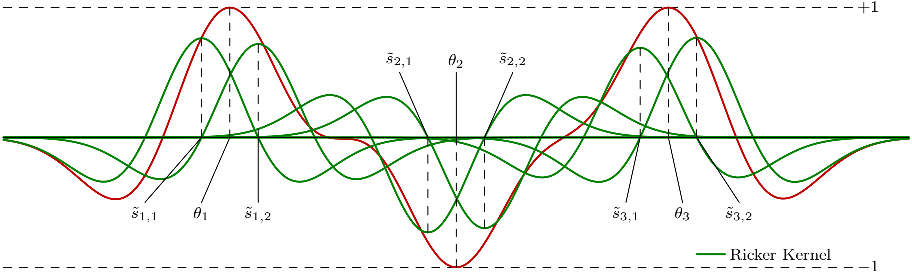
Fit v so that for all $\theta_i \in T$

$$Q(\theta_i) = \text{sign}(c_i)$$
$$Q'(\theta_i) = 0$$

It works!


+1
$\tilde{s}_{2,1}$
θ_2
$\tilde{s}_{2,2}$
$\tilde{s}_{1,1}$
θ_1
$\tilde{s}_{1,2}$
$\tilde{s}_{3,1}$
θ_3
$\tilde{s}_{3,2}$
Gaussian Kernel
−1

It works!



+1
−1
$\tilde{s}_{1,1}$
θ_1
$\tilde{s}_{1,2}$
$\tilde{s}_{2,1}$
θ_2
$\tilde{s}_{2,2}$
$\tilde{s}_{3,1}$
θ_3
$\tilde{s}_{3,2}$
Ricker Kernel

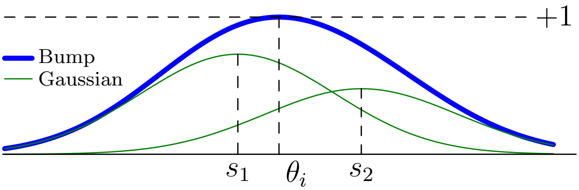
Certificate construction

Problem: The construction is difficult to analyze (coefficients vary)

Solution: Reparametrization into *bumps* and *waves*

$$
\begin{aligned}
Q(\theta) &= \sum_{s_j \in \widetilde{S}} v_j K\left(s_j - \theta\right) \\
&= \sum_{\theta_i \in T} \alpha_i B_{\theta_i}(\theta, \tilde{s}_{i,1}, \tilde{s}_{i,2}) + \beta_i W_{\theta_i}(\theta, \tilde{s}_{i,1}, \tilde{s}_{i,2}),
\end{aligned}
$$

Bump function (Gaussian kernel)

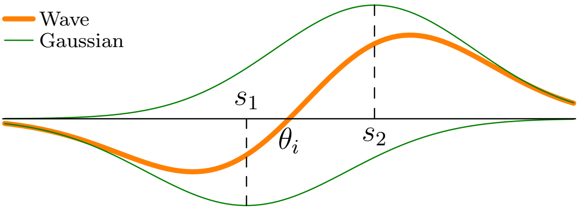


$$B_{\theta_i}(\theta, \tilde{s}_{i,1}, \tilde{s}_{i,2}) := b_{i,1} K(\tilde{s}_{i,1} - \theta) + b_{i,2} K(\tilde{s}_{i,2} - \theta)$$

$$B_{\theta_i}(\theta_i, \tilde{s}_{i,1}, \tilde{s}_{i,2}) = 1$$

$$\frac{\partial}{\partial \theta} B_{\theta_i}(\theta_i, \tilde{s}_{i,1}, \tilde{s}_{i,2}) = 0$$

Wave function (Gaussian kernel)

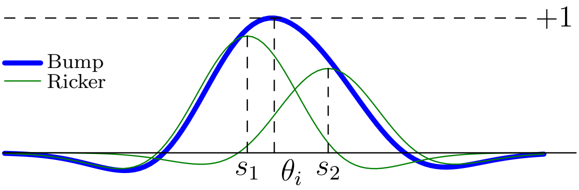


$$W_{\theta_i}(\theta, \tilde{s}_{i,1}, \tilde{s}_{i,2}) = w_{i,1} K(\tilde{s}_{i,1} - \theta) + w_{i,2} K(\tilde{s}_{i,2} - \theta)$$

$$W_{\theta_i}(\theta_i, \tilde{s}_{i,1}, \tilde{s}_{i,2}) = 0$$

$$\frac{\partial}{\partial \theta} W_{\theta_i}(\theta_i, \tilde{s}_{i,1}, \tilde{s}_{i,2}) = 1$$

Bump function (Ricker wavelet)

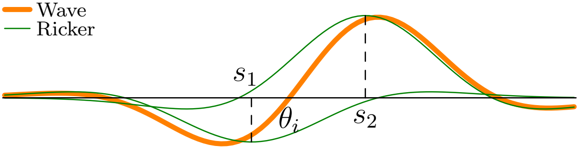


$$B_{\theta_i}(\theta, \tilde{s}_{i,1}, \tilde{s}_{i,2}) := b_{i,1} K(\tilde{s}_{i,1} - \theta) + b_{i,2} K(\tilde{s}_{i,2} - \theta)$$

$$B_{\theta_i}(\theta_i, \tilde{s}_{i,1}, \tilde{s}_{i,2}) = 1$$

$$\frac{\partial}{\partial \theta} B_{\theta_i}(\theta_i, \tilde{s}_{i,1}, \tilde{s}_{i,2}) = 0$$

Wave function (Ricker wavelet)



$$W_{\theta_i}(\theta, \tilde{s}_{i,1}, \tilde{s}_{i,2}) = w_{i,1}K(\tilde{s}_{i,1} - \theta) + w_{i,2}K(\tilde{s}_{i,2} - \theta)$$

$$W_{\theta_i}(\theta_i, \tilde{s}_{i,1}, \tilde{s}_{i,2}) = 0$$

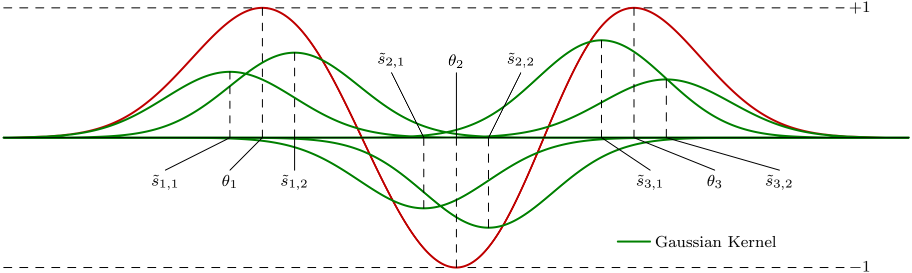
$$\frac{\partial}{\partial\theta} W_{\theta_i}(\theta_i, \tilde{s}_{i,1}, \tilde{s}_{i,2}) = 1$$

Certificate construction

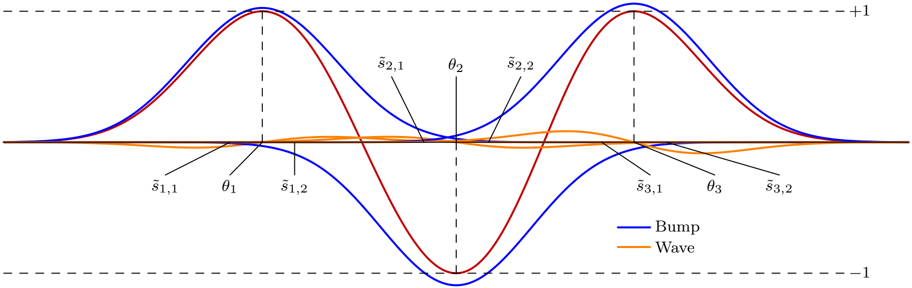
Reparametrization decouples the coefficients

$$
\begin{aligned}
Q(\theta) &= \sum_{s_j \in \widetilde{S}} v_j K\left(s_j - \theta\right) \\
&= \sum_{\theta_i \in T} \alpha_i B_{\theta_i}(\theta, \tilde{s}_{i,1}, \tilde{s}_{i,2}) + \beta_i W_{\theta_i}(\theta, \tilde{s}_{i,1}, \tilde{s}_{i,2}) \\
&\approx \sum_{\theta_i \in T} \operatorname{sign}\left(c_i\right) B_{\theta_i}(\theta, \tilde{s}_{i,1}, \tilde{s}_{i,2})
\end{aligned}
$$

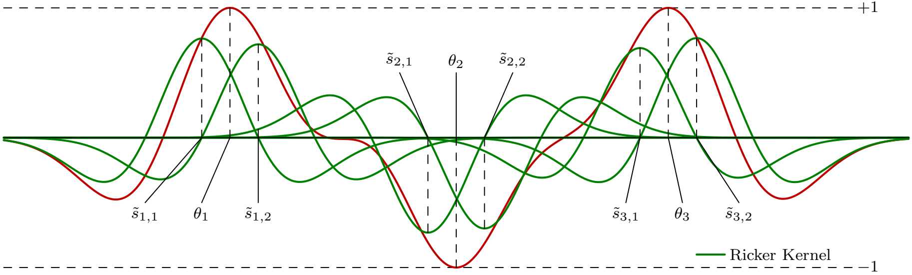
Certificate for deconvolution (Gaussian kernel)



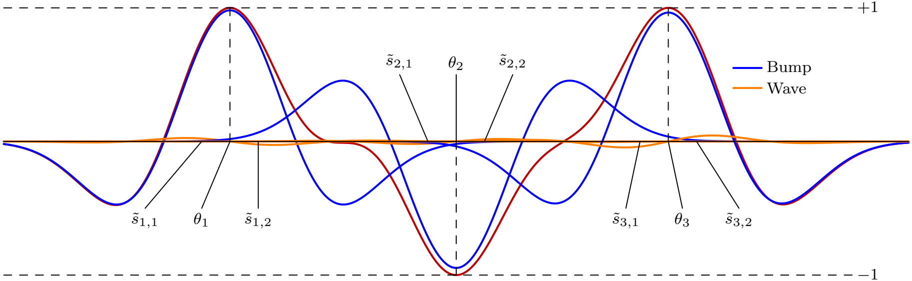
Certificate for deconvolution (Gaussian kernel)



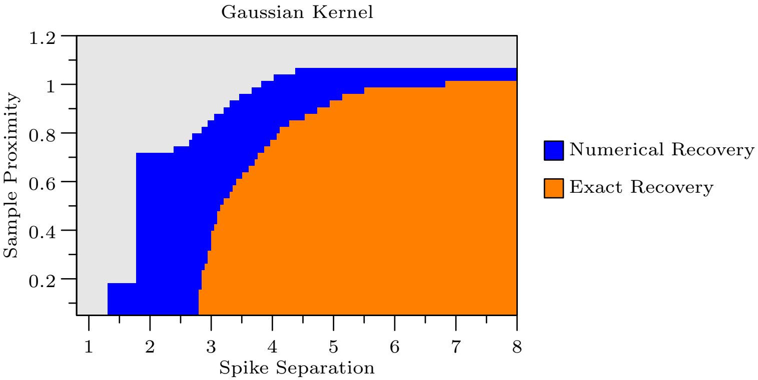
Certificate for deconvolution (Ricker wavelet)



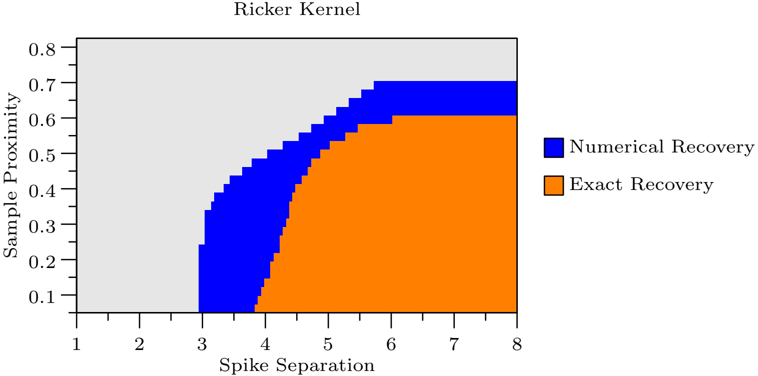
Certificate for deconvolution (Ricker wavelet)



Exact recovery guarantees [Bernstein, F. 2017]



Exact recovery guarantees [Bernstein, F. 2017]



Separable Nonlinear Inverse Problems

Sparse Recovery for SNL Problems

Super-resolution

Deconvolution

SNL Problems with Correlation Decay

SNL problems

Joint work with Brett Bernstein (Courant) and Sheng Liu (CDS, NYU)

General SNL problems

The function ϕ may not be available explicitly but can often be computed numerically by solving a differential equation

- Source localization in EEG
- Direction of arrival in radar / sonar
- Magnetic-resonance fingerprinting

Mathematical model

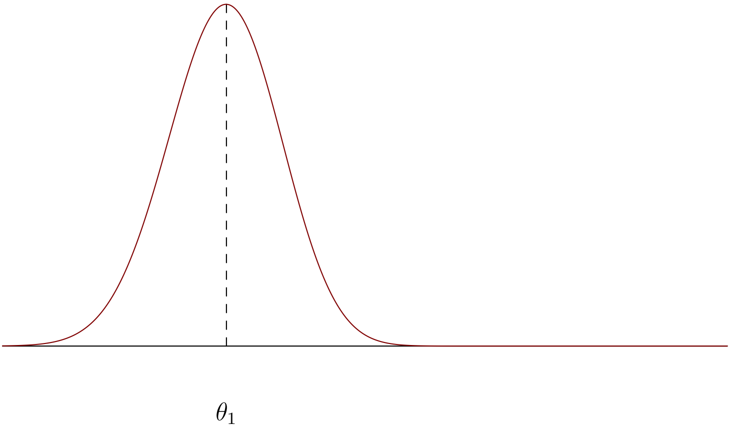
- Signal: superposition of Dirac measures with support T

$$x = \sum_j c_j \delta_{\theta_j} \qquad c_j \in \mathbb{R},\ \theta_j \in T \subset [0,1]$$

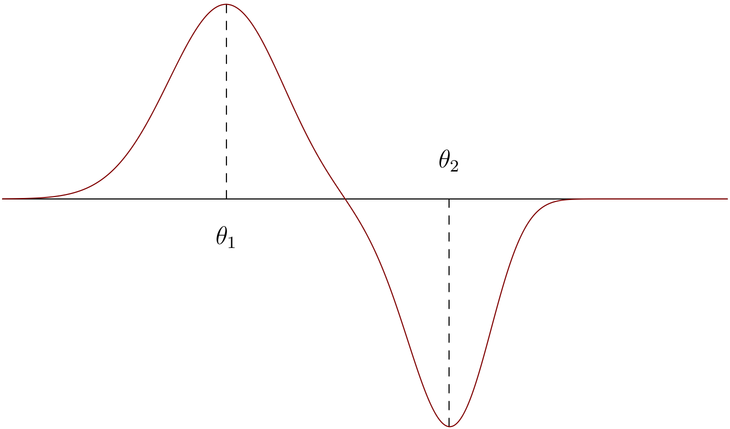
- Data: n measurements following SNL model

$$y = \int \phi(\theta)\, x(\mathrm{d}\theta)$$

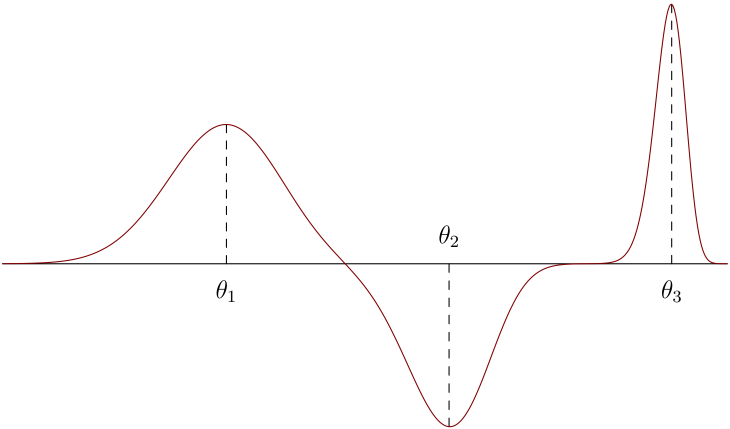
Diffusion on a rod with varying conductivity



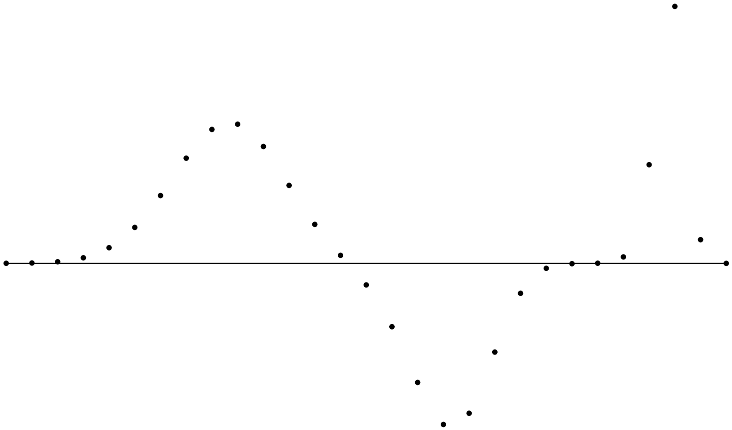
Diffusion on a rod with varying conductivity



Diffusion on a rod with varying conductivity

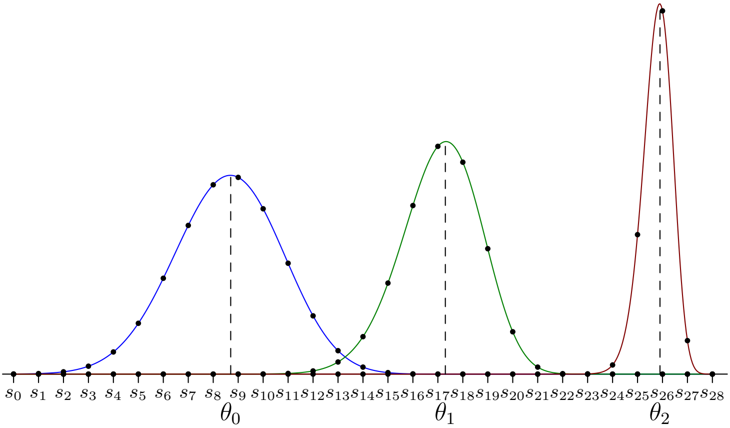


Diffusion on a rod with varying conductivity

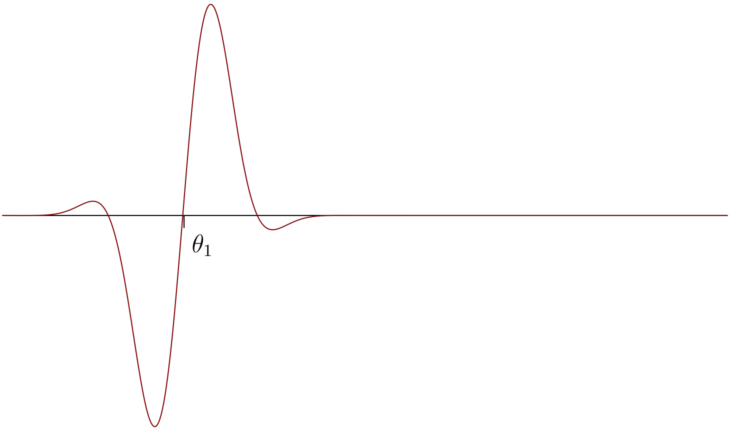


Diffusion on a rod with varying conductivity

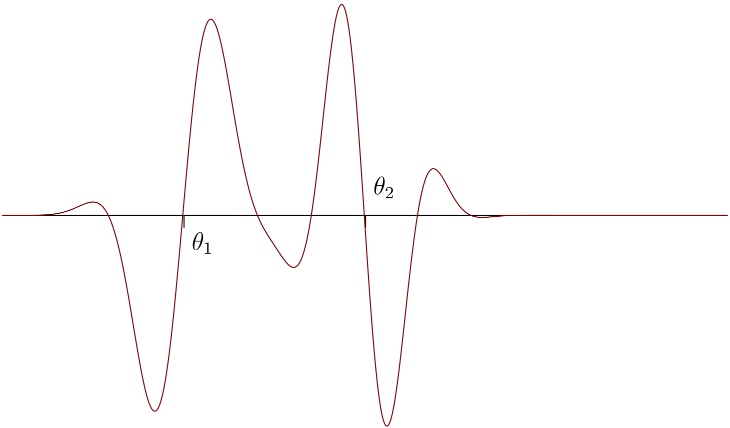
$\phi(\theta)$ can be computed by solving differential equation



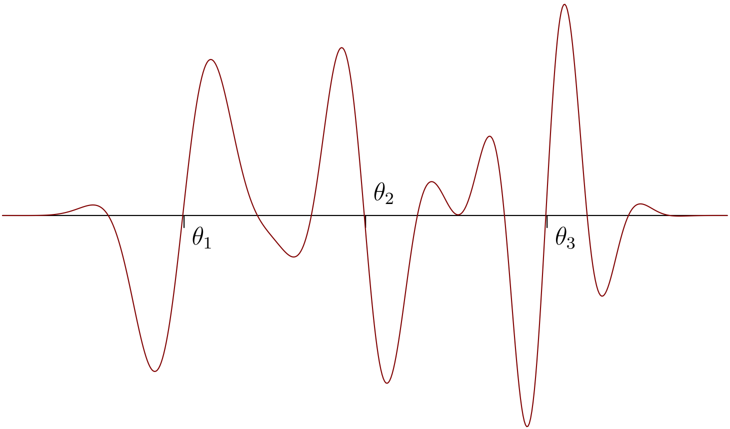
Time-frequency pulses



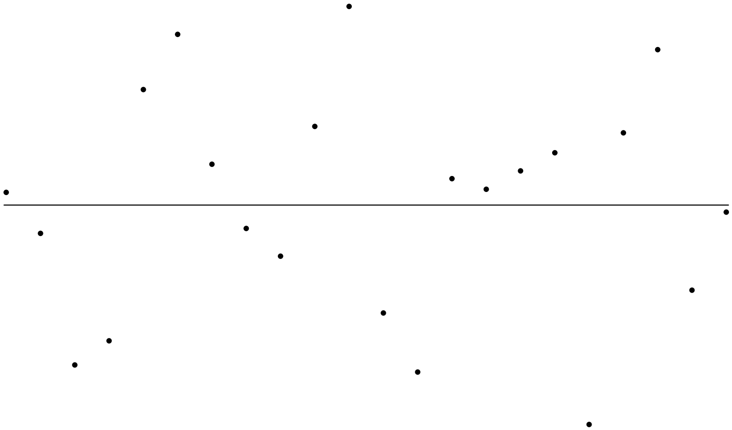
Time-frequency pulses



Time-frequency pulses



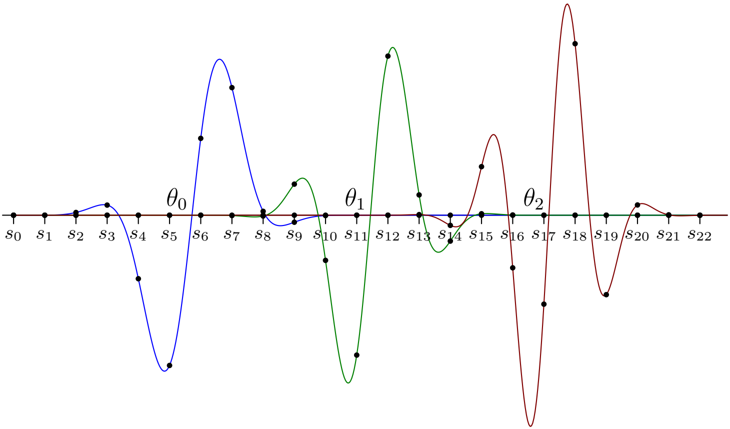
Time-frequency pulses



Time-frequency pulses

Gabor wavelets in 1D

$$\phi(\theta)_k = \exp\left(-\frac{(s_k - \theta)^2}{2\sigma}\right)\sin(150\, s_k(s_k - \theta)) \qquad \theta \in [0, 1]$$



Sparse estimation for general SNL problems

Problem: Sparse recovery requires RIP-like properties that do not hold for SNL problems with smooth ϕ (even if we discretize)

We cannot hope to recover all sparse signals

How about signals such that $\phi(\theta_i)^T \phi(\theta_j)$ is small for all $\theta_i \neq \theta_j$ in T?

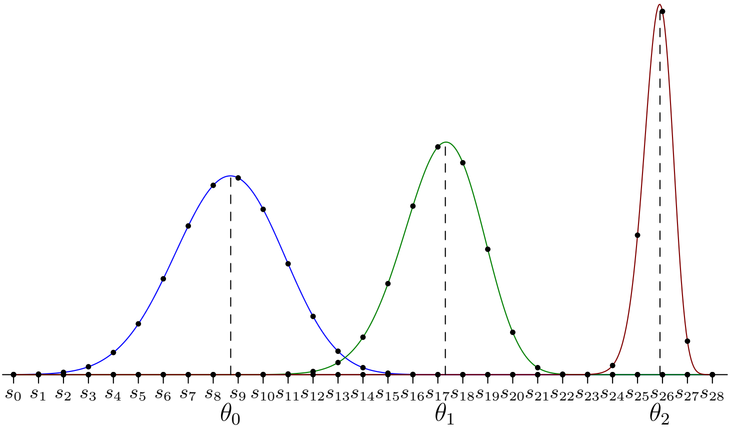
Challenge: Prove guarantees for general SNL problems that only depend on correlation structure

SNL problems with correlation decay

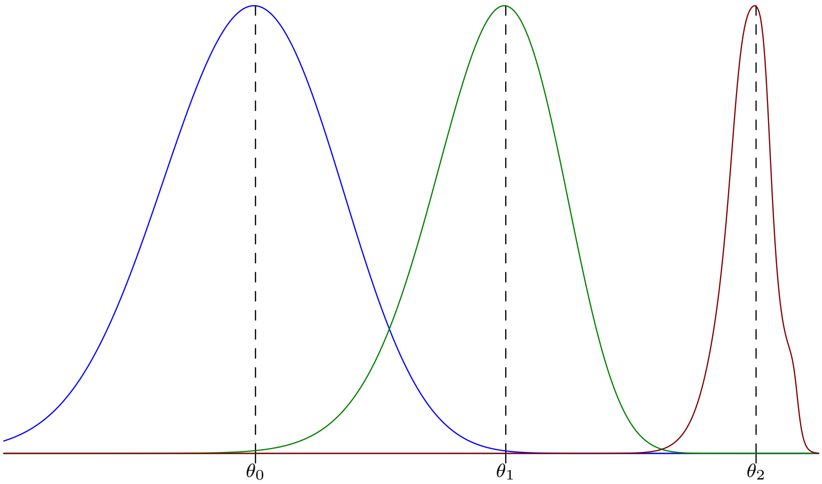
Aim: Guarantees for signals under separation conditions with respect to support-centered correlations $\rho_1, \ldots, \rho_s$

$$\rho_i(\theta) := \phi(\theta_i)^T \phi(\theta)$$

Diffusion on a rod with varying conductivity

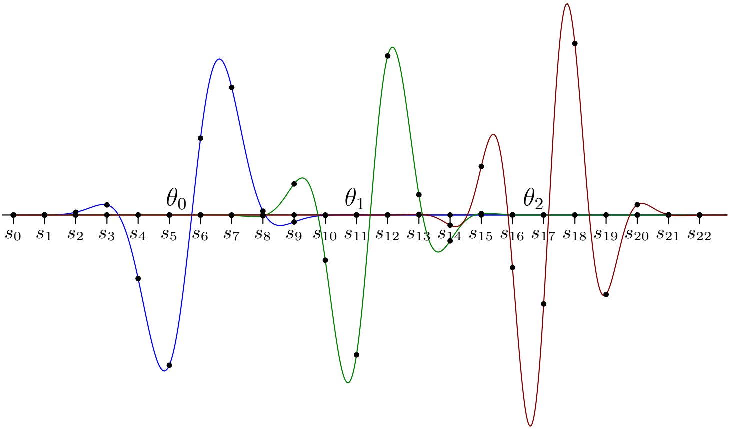


Support-centered correlations

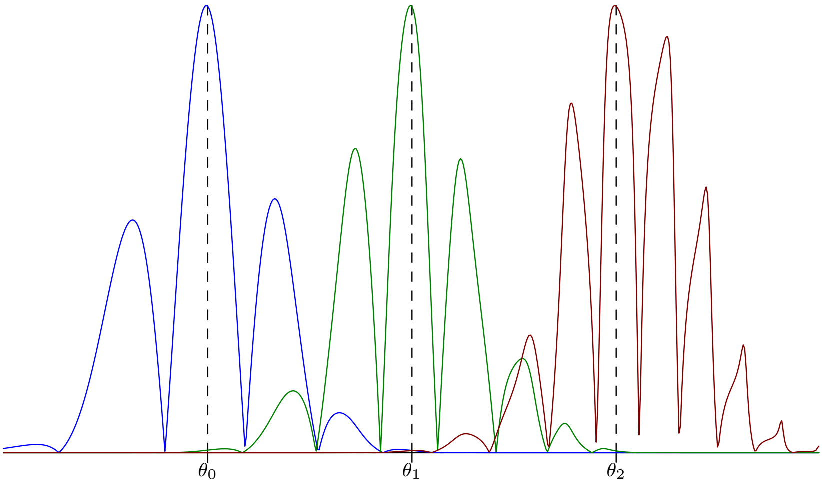


Time-frequency pulses

$$\phi(\theta)_k = \exp\left(-\frac{(s_k - \theta)^2}{2\sigma}\right)\sin(150\, s_k(s_k - \theta)) \qquad \theta \in [0,1]$$



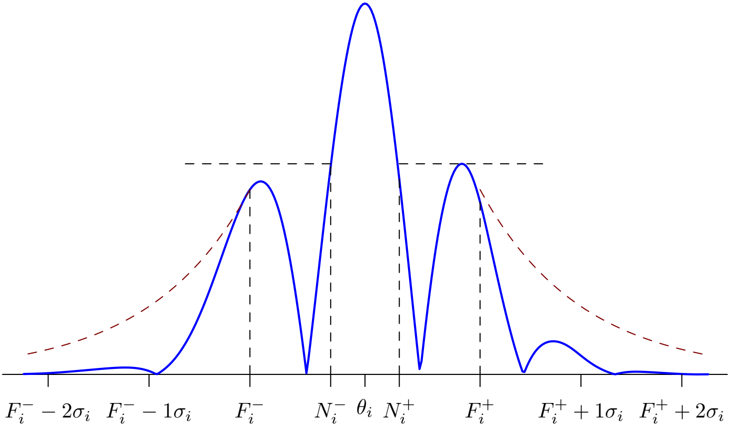
Support-centered correlations



Correlation decay

Parametrized by $F_i^- < N_i^- < N_i^+ < F_i^+$ and σ_i

Parameters can be different at each θ_i



Correlation decay

- ρ_i is concave in $[N_i^-, N_i^+]$: $\rho_i''(\theta) < -\gamma_0$
- ρ_i is bounded outside $[N_i^-, N_i^+]$: $|\rho_i(\theta)| < \gamma_1$
- ρ_i decays for $\theta < F_i^-$ and $\theta > F_i^+$:

  $|\rho_i(\theta)| < \gamma_2 e^{-(|\theta - F_i^-|)/\sigma_i}$ for $\theta < F_i^-$
  $|\rho_i(\theta)| < \gamma_2 e^{-(|\theta - F_i^+|)/\sigma_i}$ for $\theta > F_i^+$

  Choice of exponential decay is arbitrary

Correlation decay

Additional condition on correlation derivatives

$$\rho_i^{(q,r)}(\theta) := \phi^{(q)}(\theta_i)^T \phi^{(r)}(\theta)$$

$\rho_i^{(q,r)}$ decays for $\theta < F_i^-$ and $\theta > F_i^+$, for $q = 0, 1,\ r = 0, 1, 2$:

$$\left|\rho_i^{(q,r)}(\theta)\right| < \gamma_2 e^{-(|\theta - F_i^-|)/\sigma_i} \text{ for } \theta < F_i^-$$
$$\left|\rho_i^{(q,r)}(\theta)\right| < \gamma_2 e^{-(|\theta - F_i^+|)/\sigma_i} \text{ for } \theta > F_i^+$$

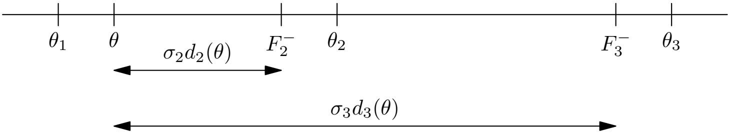
Open question: Is this necessary?

Normalized distance

Normalized distance from θ to $\theta_j > \theta$

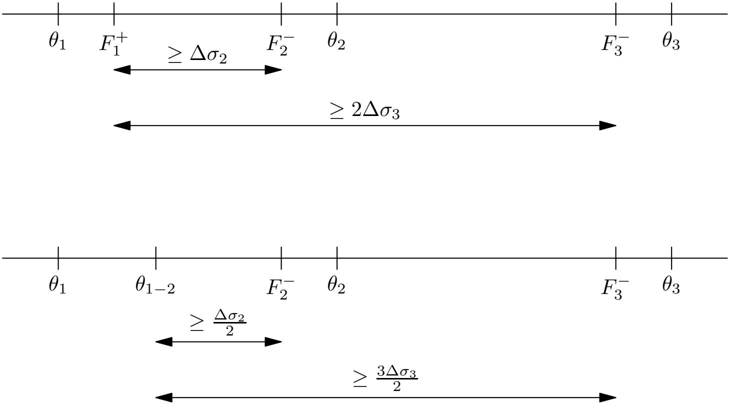
$$d_j(\theta) := \frac{F_j^- - \theta}{\sigma_j}$$

If $d_j(\theta)$ is large, $\phi(\theta)$ and $\phi(\theta_j)$ are not very correlated



Minimum separation conditions

Normalized distance between spikes is equal to Δ



Guarantees for SNL problems with decaying correlation

Theorem [Bernstein, F. 2018]

For any SNL problem with decaying correlation TV-norm minimization achieves exact recovery under the separation conditions if

$$\Delta > C$$

for a fixed constant C depending on the decay bounds γ_0, γ_1, γ_2

Dual certificate for SNL problems

$v \in \mathbb{R}^n$ is a dual certificate associated to

$$x = \sum_j c_j \delta_{\theta_j} \qquad c_j \in \mathbb{R},\ \theta_j \in T$$

if

$$Q(\theta) := v^T \phi(\theta)$$

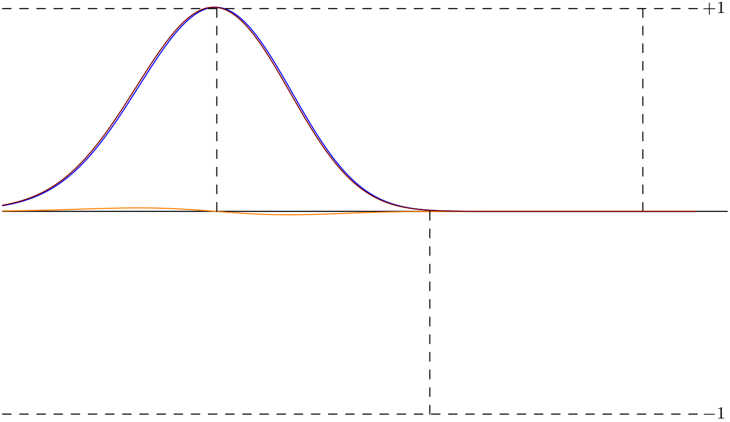
$$Q(\theta_j) = \text{sign}(c_j) \qquad \text{if } \theta_j \in T$$

$$|Q(\theta)| < 1 \qquad \text{if } \theta \notin T$$

Dual certificate construction

Use support-centered correlations to interpolate sign pattern

$$Q(\theta) := \sum_{i=1}^{s} \alpha_i \, \rho_i(\theta)$$



Dual certificate construction

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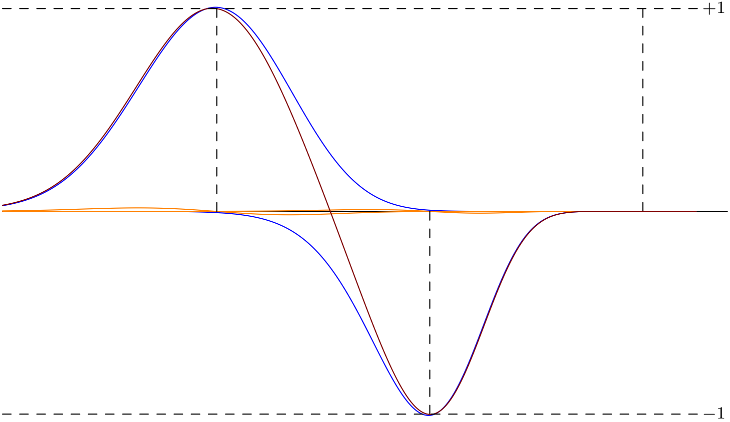
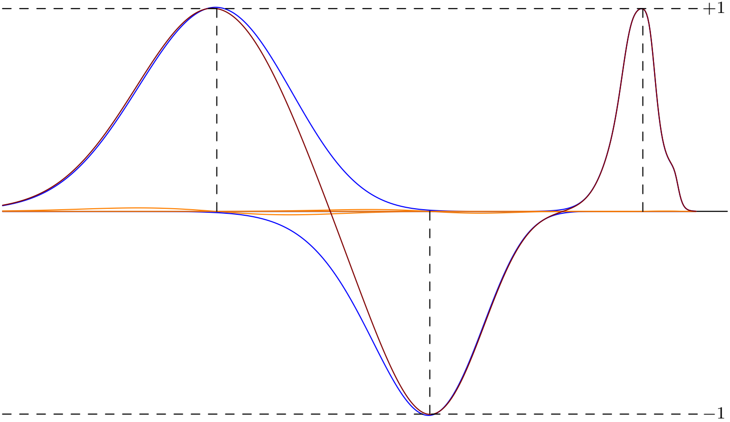


Dual certificate construction

Use support-centered correlations to interpolate sign pattern

$$Q(\theta) := \sum_{i=1}^{s} \alpha_i \, \rho_i(\theta)$$



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$$Q(\theta) := \sum_{i=1}^{s} \alpha_i \, \rho_i(\theta)$$

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$$Q(\theta) := \sum_{i=1}^{s} \alpha_i \, \rho_i(\theta) = \sum_{i=1}^{s} \alpha_i \, \phi(\theta_i)^T \phi(\theta)$$

Dual certificate construction

Use support-centered correlations to interpolate sign pattern

$$
\begin{aligned}
Q(\theta) &:= \sum_{i=1}^{s} \alpha_i \, \rho_i(\theta) \\
&= \sum_{i=1}^{s} \alpha_i \, \phi(\theta_i)^T \phi(\theta) \\
&= v^T \phi(\theta) \qquad v := \sum_{i=1}^{s} \alpha_i \, \phi(\theta_i)
\end{aligned}
$$

Dual certificate construction

Use support-centered correlations to interpolate sign pattern

$$\begin{aligned} Q(\theta) &:= \sum_{i=1}^{s} \alpha_i \, \rho_i(\theta) \\ &= \sum_{i=1}^{s} \alpha_i \, \phi(\theta_i)^T \phi(\theta) \\ &= v^T \phi(\theta) \qquad v := \sum_{i=1}^{s} \alpha_i \, \phi(\theta_i) \end{aligned}$$

Technical detail: Correction term to ensure derivative vanishes

Robustness to noise / outliers

Variations of dual certificates establish robustness at small noise levels (Candès, F. 2013), (F. 2013), (Bernstein, F. 2017)

Exact recovery with constant number of outliers (up to log factors) (F., Tang, Wang, Zheng 2017), (Bernstein, F. 2017)

Open questions: Analysis of higher-noise levels and discretization error, robustness for positive amplitudes

Drawbacks

Solving convex program is computationally expensive

Approach doesn't scale well at high dimensions

In practice, reweighting is need to obtain sparse solutions for noisy data

Open question: Analysis of other techniques (reweighting methods, descent methods on nonconvex cost functions)

Conclusion

Previous works focus mostly on random operators

For deterministic problems sparsity is not enough!

Under separation conditions:

1. Sharp guarantees for super-resolution and deconvolution
2. General guarantees for SNL problems with correlation decay

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